Recall :-

Θ - parameter space $\quad \Theta_0 \subseteq \Theta$

- a hypothesis is a statement about the population parameter

(null) $H_0 : \theta \in \Theta_0$

(alternative) $H_1 : \theta \in \Theta_0^c$

Example: $X_1, \ldots, X_{16} \sim \text{Normal}(\mu, 9)$ $\quad \bar{X}_{16} = 10.2$

$H_0 : \mu = 9.5$

$H_1 : \mu \neq 9.5$

$\Theta = \mathbb{R} \quad \Theta_0 = 9.5$

$$\frac{\bar{X}_{16} - 9.5}{3/4} \geq c$$

Method for finding tests

θ can be Vector

Likelihood Ratio test (LRT)

Let $X_1, \dots, X_n$ be iid with pmf or pdf "$f(x|\theta)$"

Likelihood function of θ

$$L(\theta | x_1, \dots, x_n) = \prod_{i=1}^{n} f(x_i | \theta)$$

In short form $\underset{\sim}{x} = (x_1, \dots, x_n)$ $\quad L(\theta|x) = \prod_{i=1}^{n} f(x_i|\theta)$ "=" $f(\underset{\sim}{x}|\theta)$

Definition 8.2.1 :- The likelihood ratio test statistic (LRT)

for the test $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_0^c$

is given by

$$\lambda(\underset{\sim}{x}) = \frac{\sup\limits_{\theta \in \Theta_0} L(\theta | x_1 \ldots x_n)}{\sup\limits_{\theta \in \Theta} L(\theta | x_1 \ldots x_n)}$$

Reject if H_0 if $x_1 \ldots x_n$ is such that

$\underset{\sim}{x} = (x_1, \ldots x_n)$ $\qquad \lambda(\underset{\sim}{x}) \leq c$ for some suitable $c \in [0,1]$

How to select c?

Understanding LRT

- **Numerator of $\lambda(\underset{\sim}{x})$** is maximum probability of the observed sample with maximum computed over parameters that are in H_0.

  "$\sup_{\theta \in \Theta_0} L(\theta|\underset{\sim}{x})$"

- **Denominator of $\lambda(\underset{\sim}{x})$** is maximum probability of the sample over all values of the parameter

- Ratio $\equiv \lambda(\underset{\sim}{x})$ - will be small when the maximum of $L(\theta|\underset{\sim}{x})$ occurs in the parameters which fall in H_1

  $\therefore$ Therefore LRT rejects H_0 in such cases.

Example 8.2.2

$X_1, \dots, X_n$ i.i.d sample from Normal $(\theta, 1)$

$H_0 . \ \theta = \theta_0$ vs $H_1 . \ \theta \neq \theta_0$

Numerator of $\lambda(\underset{\sim}{x})$:- $\sup_{\theta \in \Theta_0} L(\theta|\underset{\sim}{x}) = L(\theta_0|\underset{\sim}{x}) = \prod_{i=1}^{n} f(x_i|\theta_0) = \prod_{i=1}^{n} \frac{e^{-\frac{(x_i-\theta_0)^2}{2}}}{\sqrt{2\pi}}$

Denominator of $\lambda(\underset{\sim}{x})$:- $\sup_{\theta \in \Theta} L(\theta|\underset{\sim}{x}) = \sup_{\theta \in \mathbb{R}} L(\theta|\underset{\sim}{x}) = \sup_{\theta \in \mathbb{R}} \prod_{i=1}^{n} \frac{e^{-\frac{(x_i-\theta)^2}{2}}}{\sqrt{2\pi}}$

$$\boxed{\lambda(\underset{\sim}{x}) = \frac{e^{-\sum_{i=1}^{n}(x_i-\theta_0)^2}}{\sup_{\theta \in \mathbb{R}} e^{-\sum_{i=1}^{n}(x_i-\theta)^2}}}$$

$\boxed{\text{LRT: Reject if } \lambda(\underset{\sim}{x}) \leq c}$

check :-

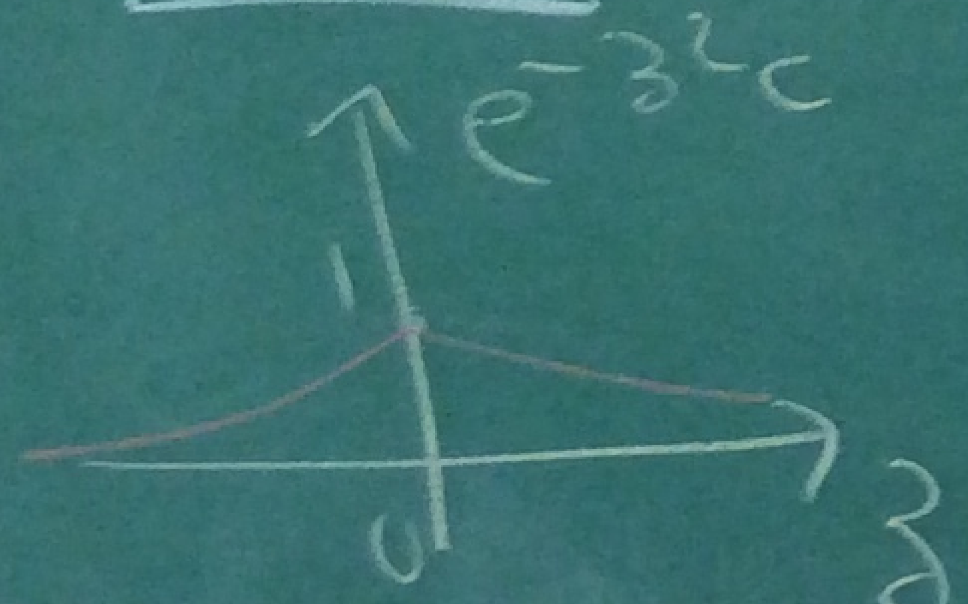
$\lambda($

ob

Check :- $\displaystyle\sum_{i=1}^{n}(x_i-\theta)^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\theta)^2$

$$\lambda(\underset{\sim}{x}) = \frac{e^{-\cancel{\left(\sum_{i=1}^{n}(x_i-\bar{x})^2/2\right)}}\; e^{-\frac{n(\bar{x}-\theta_0)^2}{2}}}{\sup\limits_{\theta\in\mathbb{R}}\left[e^{-\cancel{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{2}}}\right]\left[e^{-\frac{n(\bar{x}-\theta)^2}{2}}\right]} = \frac{e^{-\frac{n(\bar{x}-\theta_0)^2}{2}}}{\sup\limits_{\theta\in\mathbb{R}} e^{-\frac{n(\bar{x}-\theta)^2}{2}}}$$

observe : $\sup\limits_{\theta\in\mathbb{R}} e^{-\frac{n(\bar{x}-\theta)^2}{2}} = 1$, $\therefore$ $\boxed{\lambda(\underset{\sim}{x}) = e^{-\frac{n(\bar{x}-\theta_0)^2}{2}}}$



Recap. $X_1 \ldots X_n$ i.i.d $\sim$ Normal$(\theta, 1)$

$H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$

$\begin{pmatrix} x_1 \ldots x_n \\ \text{sample} \end{pmatrix}$ Reject if $\lambda(\underset{\sim}{x}) \leq c$ $\quad c \in [0,1]$
(LRT)

i.e. $e^{-\frac{n(\bar{x}-\theta_0)^2}{2}} \leq c$

i.e. $-\frac{n(\bar{x}-\theta_0)^2}{2} \leq \log c$, i.e. $n(\bar{x}-\theta_0)^2 \geq -2\log c$

i.e. Reject H_0 if $\frac{\sqrt{n}|\bar{x}-\theta_0|}{1} > \sqrt{-2\log c}$ $\equiv$ "Same as 2-sided z-test"

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8.3 Error Probabilities & Power function

A test for $\begin{array}{l} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta_0^c \end{array}$ might make

two types of errors.

Type-I ERROR : If $\theta \in \Theta_0$ but the test (incorrectly) decides to reject H_0

Type II Error :- If $\theta \in \Theta_0^c$ but the test (incorrectly) decides to accept H_0

TRUTH \ DECISION	H_0	H_1
H_0	✓	TYPE I ERROR
H_1	TYPE II ERROR	✓

test (incorrectly)

H_0

R - Rejection region

If $\theta \in \Theta_0$ then $P_\theta(X \in R) \equiv$ Probability of Type I Error

[Probability that we reject H_0 when th true]

If $\theta \in \Theta_0^c$ then $P_\theta(X \in R^c) \equiv$ Probability of Type II Error

[Probability that we accept H_0 when the alternative is true]

Example 8.3.2

$X \sim \text{Binomial}(5, \theta)$

$H_0: \theta \leq \frac{1}{2}$

$H_1: \theta > \frac{1}{2}$

Test 1 :- Toss a coin five times & reject H_0 if all 5 are heads.

Power function :- $\beta(\theta) = P_\theta(X \in R) = P_\theta(X=5) = \theta^5$

If $\theta \in \Theta_0$, ie $\theta \le \frac{1}{2} \Rightarrow \beta(\theta) = \theta^5 \le \left(\frac{1}{2}\right)^5$

✓ Probability of Type I error is at most $\left(\frac{1}{2}\right)^5$

$= 0.0316$

If $\theta \in \Theta_0^c$, ie $\theta > \frac{1}{2} \Rightarrow 1-\beta(\theta) = 1-\theta^5 \le 1-\left(\frac{1}{2}\right)^5$

$= 1 - 0.03 = 0.9\ldots$

✓ Probability of Type II error can be "0.9" as large as

$P(\text{Type II error}) < \frac{1}{2} \Leftrightarrow 1-\theta^5 < \frac{1}{2}$

$\Leftrightarrow \theta^5 > \frac{1}{2}$

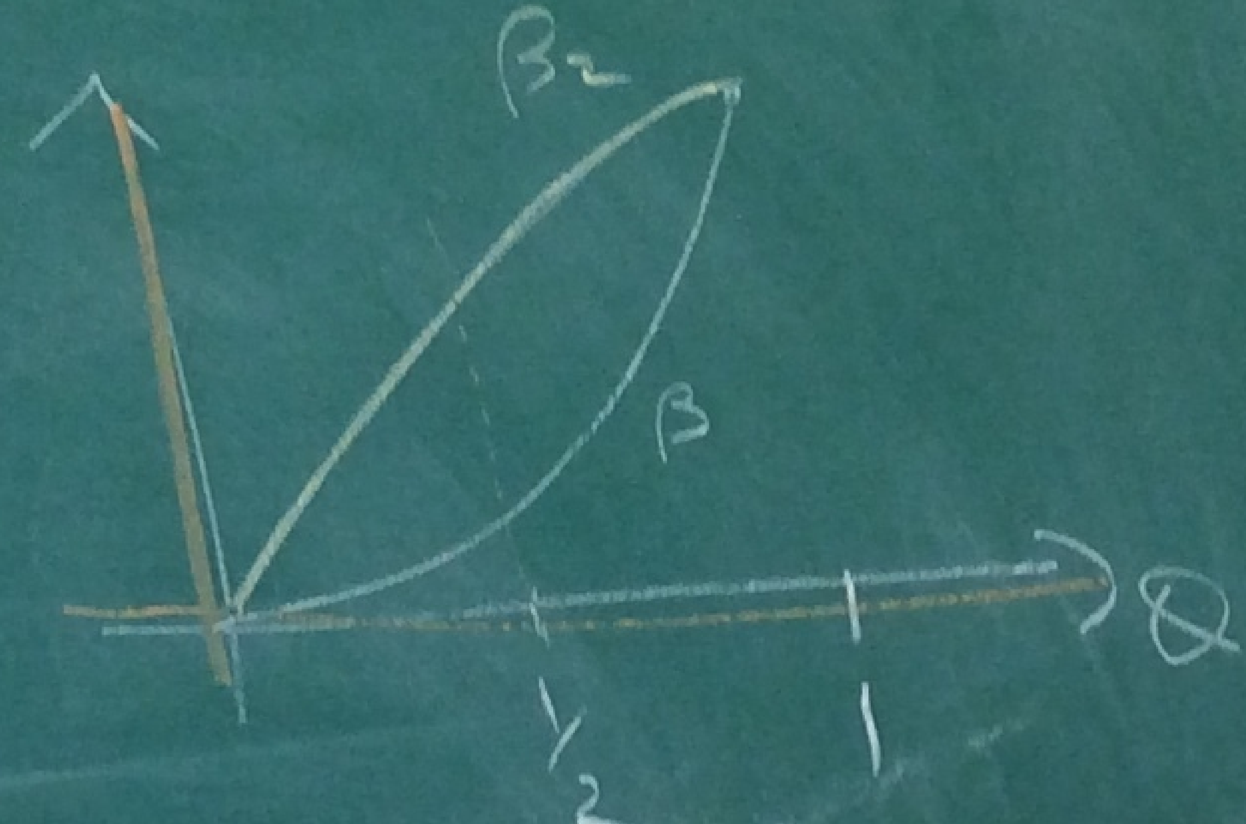
$\Leftrightarrow \theta > (1/2)^{1/5} = 0.87$

Test II

Reject if $X = 3, 4$ or 5

$$\beta_2(\theta) = P_\theta(X = 3, 4 \text{ or } 5) = {}^5C_3\, \theta^3 (1-\theta)^2 + {}^5C_4\, \theta^4 (1-\theta)^1 + \theta^5$$

Check :- Probability of Type I error is larger but Probability of Type II error is smaller



> In such cases the researcher must decide on a suitable error structure

Example 8.3.3 $X_1, X_2, \ldots X_n$ iid Normal $(\theta, 1)$

$$H_0 : \theta \leq \theta_0 \quad vs \quad H_1 : \theta > \theta_0$$

Test: Reject H_0 if $\dfrac{\sqrt{n}\,(\bar{X} - \theta_0)}{1} \geq c$

$$\beta(\theta) = P_\theta(X \in R) = P_\theta(\sqrt{n}\,(\bar{X} - \theta_0) \geq c)$$

$$= P_\theta(\sqrt{n}(\bar{X}) \geq c + \sqrt{n}\,\theta_0)$$

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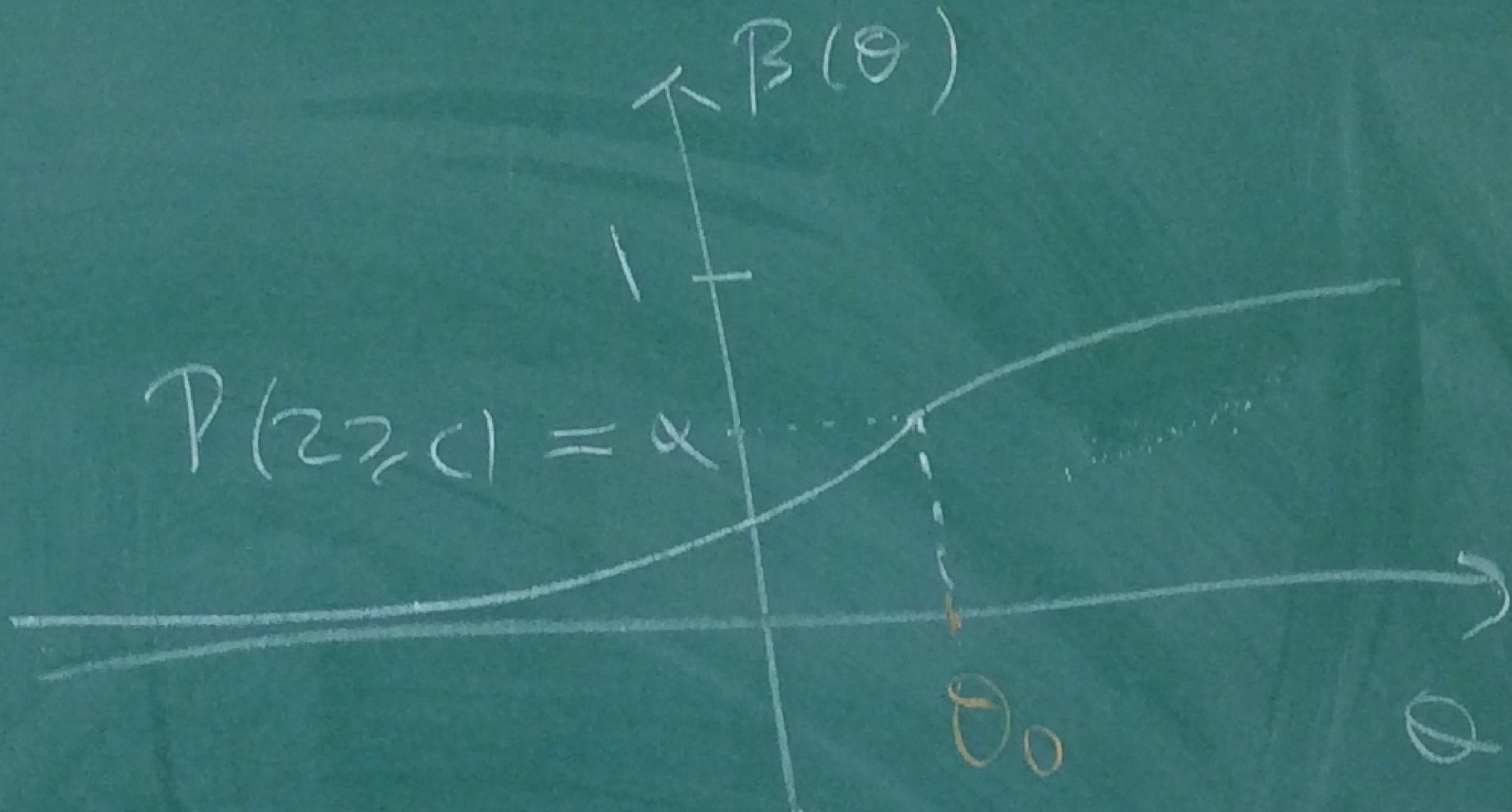
$$= \mathbb{P}_\theta\left(\sqrt{n}(\bar{X} - \theta) \geq c + \sqrt{n}\,\theta_0 - \sqrt{n}\,\theta\right)$$

$$= P\left(Z \geq c + \sqrt{n}(\theta_0 - \theta)\right)$$

where $Z \sim \text{Normal}(0,1)$



$$\alpha = P(Z \geq c)$$

$$\beta(\theta_0) = \alpha$$

$$\lim_{\theta \to -\infty} \beta(\theta) = 0$$

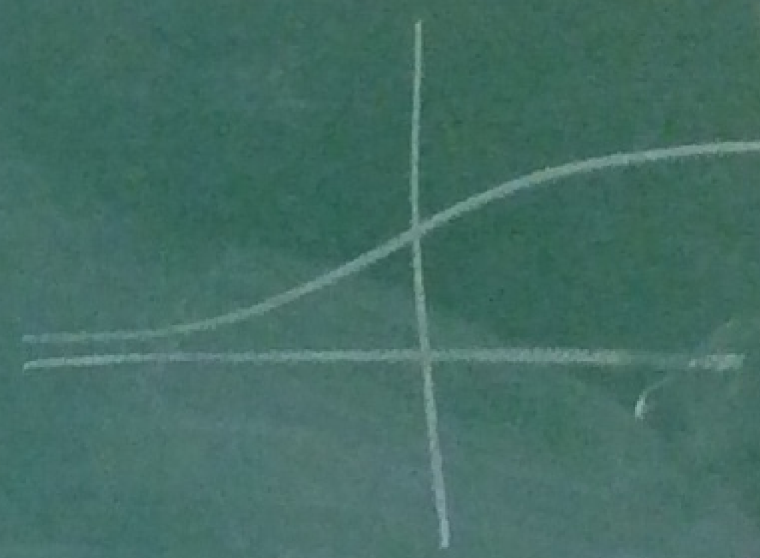
$$\lim_{\theta \to \infty} \beta(\theta) = 1$$

Typically the power function will depend on the sample size

Example 8.33 (continued)

- We want Probability of Type I error to be atmost 0.1
- If $\theta \geq \theta_0 + 1$ then we want Probability of Type II error to be at most 0.2

$$\beta(\theta) = P(Z \geq c + \sqrt{n}(\theta_0 - \theta))$$



$\theta \in \Theta_0$, ie $\theta \le \theta_0$ need $\beta(\theta) \le 0.1$

$\theta > \theta_0 + 1$, need $1 - \beta(\theta) \le 0.2$

As $\beta(\cdot)$ is increasing function, all we need is

$$\beta(\theta_0) = 0.1 \quad (\Rightarrow \theta \le \theta_0 \;\; \beta(\theta) \le 0.1)$$

$$1 - \beta(\theta_0 + 1) = 0.2 \quad (\Rightarrow \theta > \theta_0 + 1, \; 1 - \beta(\theta) \le 0.2)$$

$\beta(\theta_0) = P(Z \ge c) = 0.1 \Leftrightarrow c = 1.28$ AND

$$1-\beta(\theta_0+1) = 1 - P(Z \geq c-\sqrt{n}) = 0.2$$

ie $P(Z \geq c-\sqrt{n}) = 0.8$

$\iff c-\sqrt{n} = -0.84$

$\iff 1.28-\sqrt{n} = -0.84$

$\iff n = 4.49$

<u>Summary</u> $X_1 \ldots X_n \sim \text{Normal}(\theta, 1)$

$H_0: \theta \leq \theta_0$ then Reject H_0 if $\sqrt{n}(\bar{X}-\theta_0) \geq c$

$H_1: \theta > \theta_0$

$$\beta(\theta) = P\left(Z \geq c + \sqrt{n}\,(\theta_0 - \theta)\right)$$

If we want P(Type I error) at most 0.1

& P(Type II error) at most 0.2 when $\theta > \theta_0 + 1$

then we should take $c = 1.28$ & $n = 5$

Definition 8.3.5 Let $0 \leq \alpha \leq 1$. A test with power function $\beta(\theta)$ is a size α-test if $\sup_{\theta \in \Theta_0} \beta(\theta) = \alpha$