INTRODUCTION TO KPZ

JEREMY QUASTEL

Contents

1. A physical introduction

1.1. **KPZ/Stochastic Burgers/Scaling exponent.** The Kardar-Parisi-Zhang (KPZ) equation is,

$$\partial_t h = -\lambda(\partial_x h)^2 + \nu \partial_x^2 h + \sqrt{D}\xi \tag{1}$$

where ξ denotes space-time white noise which is the distribution valued Gaussian field with correlation function

$$\langle \xi(t,x), \xi(s,y) \rangle = \delta(t-s)\delta(x-y). \tag{2}$$

It is an equation for a randomly evolving height function $h \in \mathbb{R}$ which depends on position $x \in \mathbb{R}$ and time $t \in \mathbb{R}_+$. λ, ν and D are physical constants.

The equation was introduced by Kardar, Parisi and Zhang in 1986 [KPZ86], and quickly became the default model for random interface growth in physics. Mathematically, the non-linearity was too large to handle by existing methods of stochastic partial differential equations. So there is a very serious problem of well-posedness.

Formally, it is equivalent to the stochastic Burgers equation

$$\partial_t u = -\lambda \partial_x u^2 + \nu \partial_x^2 u + \sqrt{D}\partial_x \xi. \tag{3}$$

which, if things were nice, would be satisfied by $u = \partial_x h$.

An analogous equation can be written in higher dimensions,

$$\partial_t h = -\lambda |\nabla h|^2 + \nu \Delta h + \sqrt{D}\xi \tag{4}$$

with $x \in \mathbb{R}^d$. One can also attempt to generalize the non-linearity

$$\partial_t h = F(\nabla h) + \nu \Delta h + \sqrt{D}\xi \tag{5}$$

It appears that with space-time white noise forcing, *only* (1) yields a non-trivial field. We will restrict ourselves here to the one space dimension with the quadratic non-linearity (1). Even in this $1+1$ dimensional situation we are still in the very difficult case of a field theory with broken time reversible invariance.

This stochastic Burgers equation is a toy model for turbulence. A dynamical renormalization group analysis was performed in 1977, by Forster, Nelson and Stephen [FNS77] (see also [KPZ86], [vB85]) , predicting a dynamical scaling exponent

$$z = 3/2. \tag{6}$$

For the solution h of the KPZ equation this means that one expects non-trivial fluctuation behaviour under the rescaling

$$h_\epsilon(t,x) = \epsilon^{1/2} h(\epsilon^{-z}t, \epsilon^{-1}x). \tag{7}$$

We will discuss this much more precisely later in the notes.

Now we move to the physical derivation of the process and the physical predictions. Note that this entire introduction is not intended to be rigorous, but just to sketch the physical background. From a mathematical point of view, one is of course interested in proving existence and uniqueness for the equation. On the other hand, we will see that the solutions can be written in terms of a classically well-posed stochastic partial differential equation. So the main issue for these notes will be on the actual behaviour of solutions, the scaling exponents, and the large scale fluctuation behaviour which is conjectured to be universal within what is called the KPZ universality class. For this reason we will spend in Section 1 a great deal of time discussing the physical picture. The mathematics begins in Section 2.

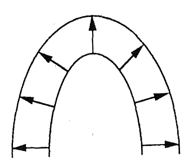
1.2. **Physical derivation.** h grows by random deposition as well as diffusion. The change in time has three contributions: 1. Slope dependent, or lateral growth, 2 . Relaxation 3. Random forcing. The equation then reads

$$\partial_t h = -\lambda F(\partial_x h) + \nu \partial_x^2 h + \sqrt{D}\xi \tag{8}$$

The $\partial_x^2 h$ term, 2, represents the simplest possible form of relaxation/smoothing/diffusion. ν is the diffusivity, or viscosity. The random forcing, 3, is assumed to be roughly independent at different positions and different times. The simplest model is Gaussian space-time white noise, which has mean zero and space-time correlations

$$\langle \xi(t,x), \xi(s,y)\rangle := E[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y). \tag{9}$$

$\sqrt{D}$ represents the strength of the noise. Traditionally it has a square root so that D is the mean square, or variance. The key term, 1, the deterministic part of the growth, is assumed to be a function only of the slope, and to be a symmetric function. Here is a picture of what we mean by lateral growth



From the picture, the natural choice for F might be $(1 + |\partial_x h|^2)^{-1/2}$, however this leads to a seemingly intractable equation. In fact what is done is to take a general F and expand

$$F(s) = F(0) + F'(0)s + \tfrac{1}{2}F''(0)s^2 + \cdots \tag{10}$$

The first term can be removed from the equation by a time shift. The second should vanish because of the symmetry, but could anyway have been removed from the equation by a constant velocity shift of coordinates. Thus the quadratic term is the first nontrivial contribution, and it is the only one kept. We arrive at

$$\partial_t h = -\lambda(\partial_x h)^2 + \nu \partial_x^2 h + \sqrt{D}\xi. \tag{11}$$

There is something wrong with this derivation. The problem is that $|\partial_x h|^2$ is not small. As we will see in 1.6, it is huge. So one needs to subtract a huge term reflecting the small scale fluctuations. Amazingly, through such a naive derivation, one finds a non-trivial field (see [KS92], [HHZ95], [BS95] for introductions.)

1.3. **Scaling.** We can restrict attention to the special choice $\lambda = \nu = \frac{1}{2}$, $D = 1$, because if h satisfies

$$\partial_t h = -\tfrac{1}{2}(\partial_x h)^2 + \tfrac{1}{2}\partial_x^2 h + \xi \tag{12}$$

then we can write

$$h_\epsilon(t,x) = \epsilon^\beta h(\epsilon^{-z}t, \epsilon^{-1}x) \tag{13}$$

and we have $\partial_t h = \epsilon^{z-\beta}\partial_t h_\epsilon$, $\partial_x h = \epsilon^{1-\beta}\partial_x h_\epsilon$ and $\partial_x^2 h = \epsilon^{2-\beta}\partial_x^2 h_\epsilon$. More interesting is how the white noise rescales,

$$\xi(t,x) \stackrel{\text{dist}}{=} \epsilon^{\frac{z+1}{2}}\xi(\epsilon^z t, \epsilon^1 x), \tag{14}$$

where the equality is in the sense that the two random fields have the same distribution. This leads to

$$\partial_t h_\epsilon = -\tfrac{1}{2}\epsilon^{2-z-\beta}(\partial_x h_\epsilon)^2 + \tfrac{1}{2}\epsilon^{2-z}\partial_x^2 h_\epsilon + \epsilon^{\beta-\frac{1}{2}z+\frac{1}{2}}\xi. \tag{15}$$

Clearly we can now choose $\lambda = \frac{1}{2}\epsilon^{2-z-\beta}$, $\nu = \frac{1}{2}\epsilon^{2-z}$, $\sqrt{D} = \epsilon^{\beta-\frac{1}{2}z+\frac{1}{2}}$ to get (1) from (12). When comparing discrete models to KPZ, one has to identify the appropriate λ, ν and D (see [Spo12] for a discussion.)

1.4. **Formal invariance of Brownian motion.** Linearizing (11) one obtains the Langevin equation,

$$\partial_t h = \nu\partial_x^2 h + \sqrt{D}\xi \tag{16}$$

whose solution is the infinite dimensional Ornstein-Uhlenbeck process,

$$h(t,x) = \int_{\mathbb{R}} p_\nu(t, x-y)h(0,y)dy + \sqrt{D}\int_0^t\int_{\mathbb{R}} p_\nu(t-s, x-y)\xi(s,y)dyds \tag{17}$$

where

$$p_\nu(t,x) = \frac{1}{\sqrt{4\pi\nu t}}e^{-x^2/4\nu t}. \tag{18}$$

Two sided Brownian motion $B(x)$, $x \in \mathbb{R}$ normalized to have

$$E[(B(y) - B(x))^2] = (2\nu)^{-1}D|y-x| \tag{19}$$

can be checked to be invariant for this process.

Remarkably, the two-sided Brownian motion is almost invariant for (11) as well. The only sticky point is that the two sided Brownian motion $B(x)$ will have a global height shift as time proceeds. In particular, what is true is that the probability measure corresponding to the distributional derivative of the Brownian motion $B'(x)$ (which is another white noise) is invariant for the stochastic Burgers equation (3). Or at the level of KPZ, the measure corresponding to $B(x) + N$, where N is given by Lebesgue measure, is invariant for (11). Note, however, that the latter is not a probability measure (it is "non-normalizable"). What we mean is that the product measure of white noise and Lebesgue measure on $\mathbb{R}$ is invariant for the process $(h'(x), h(0))$. Remember that when we say Brownian motion is almost invariant, we mean that if $h_0(x)$ is a two sided Brownian motion, then, for fixed t, $h(t,x)$ is another two sided Brownian motion in x (plus a height shift). It is not the same Brownian motion. It is a new Brownian motion correlated with the first one in an extremely non-trivial way that is not understood (except the asymptotic correlations are known. See Section 2.2.)

At a completely formal level the invariance argument proceeds as follows. We work at the level of the stochastic Burgers equation on a large circle $[-L, L)$ with $L = -L$. Set $\lambda = \nu = \frac{1}{2}$, $D = 1$. As we pointed out, the Langevin equation $\partial_t u = \frac{1}{2}\partial_x^2 u + \partial_x \xi$ preserves white noise. So consider the Burgers flow $\partial_t u = \frac{1}{2}\partial_x(u^2)$. u lives in a space of rough functions $\mathscr{R}$ which could be $H_{-1/2-}$. Let f be a nice function on $\mathscr{R}$. We hope to show that under the Burgers flow

$$\partial_t \int f(u(t))e^{-\frac{\sigma^2}{2}\int u^2} = 0 \tag{20}$$

where the integral is over $\mathscr{R}$ with respect to white noise with

$$E[u(t,x)u(s,y)] = \sigma^2\delta(t-s)\delta(x-y) \tag{21}$$

which we write formally as $e^{-\frac{\sigma^2}{2}\int u^2}$. Differentiating we obtain

$$\partial_t \int f(u(t))e^{-\frac{\sigma^2}{2}\int u^2} = \int \langle \frac{\delta f}{\delta u}, \partial_t u\rangle e^{-\frac{\sigma^2}{2}\int u^2} \tag{22}$$

where $\frac{\delta f}{\delta u}$ is the functional (Frechet) derivative, and $\langle f,g\rangle = \int_{-L}^{L} fg dx$, to differentiate it from the function space integral. Now by the Burgers equation

$$\int \langle \frac{\delta f}{\delta u}, \partial_t u\rangle e^{-\frac{\sigma^2}{2}\int u^2} = \tfrac{1}{2}\int \langle \frac{\delta f}{\delta u}, \partial_x(u^2)\rangle e^{-\frac{\sigma^2}{2}\int u^2}. \tag{23}$$

Integrating by parts,

$$\tfrac{1}{2}\int \langle \frac{\delta f}{\delta u}, \partial_x(u^2)\rangle e^{-\frac{\sigma^2}{2}\int u^2} = -\tfrac{1}{2}\int f\langle \frac{\delta}{\delta u}\partial_x(u^2) e^{-\frac{\sigma^2}{2}\int u^2}\rangle. \tag{24}$$

But

$$\langle \frac{\delta}{\delta u}\partial_x(u^2)e^{-\frac{\sigma^2}{2}\int u^2}\rangle = \langle (2\partial_x u - \sigma^2 u\partial_x(u^2))\rangle e^{-\frac{\sigma^2}{2}\int u^2}. \tag{25}$$

The last term vanishes because $u\partial_x(u^2)_x = \partial_x\frac{1}{3}(u^3)$ and because of periodic boundary conditions any exact derivative integrates to zero: $\langle \partial_x f\rangle = \int_{-L}^{L}\partial_x f = 0$. This gives $\partial_t \int f(u(t))e^{-\frac{\sigma^2}{2}\int u^2} = 0$. In fact, the Burgers part of the flow preserves white noise with *any* variance parameter σ^2. The constraint $\sigma = 1$ is set by the Langevin part.

Taking $L \to \infty$ gives the result on $\mathbb{R}$. Since the standard white noise with $\sigma^2 = 1$ is invariant for the Burgers flow $\partial_t u = \frac{1}{2}\partial_x(u^2)$ as well as the Langevin dynamics $\partial_t u = \frac{1}{2}\partial_x^2 u + \partial_x\xi$ it is invariant for the combined dynamics $\partial_t u = \frac{1}{2}\partial_x(u^2) + \frac{1}{2}\partial_x^2 u + \partial_x\xi$.

In this way, one concludes formally that white noise is invariant for the stochastic Burgers equation.

While the invariance is true, the above argument is not even correct at the physical level. The main problem is that the Burgers flow $\partial_t u = \frac{1}{2}\partial_x(u^2)$ is ill-defined (except for convex initial data) because there are no characteristics telling us how to fill in rarefaction fans. If one interprets the Burgers' flow as the usual entropy solutions, i.e. as the limit as $\nu \downarrow 0$ of $\partial_t u^\nu = u^\nu\partial_x u^\nu + \nu\partial_x^2 u^\nu$, then one has the Lax-Oleinik variational formula for the solution, $u = \partial_x h$ with

$$h(t,x) = \sup_{y\in\mathbb{R}}\left\{-\frac{(x-y)^2}{2t} + h(0,y)\right\}. \tag{26}$$

Starting from $h(0,x)$ a two-sided Brownian motion, one obtains a collection of "N"'s, i.e. the integral of a collection of Dirac masses. The statistics are known exactly [FM00] (see also [Ber01], [MS10] for more general classes of solvable initial data). At any rate, the result starting with a Brownian motion is definitely *not* a new Brownian motion, though the formal argument tells you it should be.

A correct argument for the invariance is presented in Section 3.12. Alternatively, it could be obtained from the weakly asymmetric limit of the special directed polymer model with log-Gamma distribution, which turns out to have product invariant measures for its free energy [Sep10].

In general, one expects that for any initial data, no matter how smooth, the solution becomes locally Brownian at any positive time, with the same local diffusivity. (For proofs in various special cases, see [QR11], [IC11],[Hai11], [OW11].)

1.5. **Dynamic scaling exponent.** We search for a scaling

$$h_\epsilon(t,x) = \epsilon^\beta h(\epsilon^{-z}t, \epsilon^{-1}x) \tag{27}$$

under which we can hope to see something non-trivial as $\epsilon \to 0$, i.e. on large space and time scales.

Either by taking a derivative, or subtracting constants independent of x, we can ignore global height shifts. Fixing $t = 0$, the fact that the solution is locally Brownian forces us to take

$$\beta = 1/2 \tag{28}$$

to see anything non-trivial. The equation becomes

$$\partial_t h_\epsilon = -\tfrac{1}{2}\epsilon^{2-z-\beta}(\partial_x h_\epsilon)^2 + \tfrac{1}{2}\epsilon^{2-z}\partial_x^2 h_\epsilon + \epsilon^{\beta-\frac{1}{2}z+\frac{1}{2}}\xi. \tag{29}$$

To avoid divergence of the nonlinear term we are then forced to take

$$z = 3/2. \tag{30}$$

This is how one arrives at the dynamic scaling exponent 3/2.

All the fluctuation behaviour we will observe will be at this scale

$$h_\epsilon(t,x) = \epsilon^{1/2} h(\epsilon^{-3/2}t, \epsilon^{-1}x). \tag{31}$$

1.6. **Renormalization of the nonlinear term.** Now that we know that Brownian motion is an invariant measure for the KPZ equation except for the height shift, i.e. in equilibrium $h(x) = h(t,x)$ is a two-sided Brownian motion in x, we realize the equation has a big problem. The term $(\partial_x h)^2$ cannot possibly make sense. Recall the basic quadratic variation computation for Brownian motion that for a real interval $[a,b]$, then, almost surely,

$$\lim_{n\to\infty} \sum_{i=\lfloor 2^n a\rfloor}^{\lfloor 2^n b\rfloor} |h(\tfrac{i+1}{2^n}) - h(\tfrac{i}{2^n})|^2 = 2(b-a). \tag{32}$$

The 2 is just because our Brownian motions have diffusion coefficient 2 (see (19)). The problem is not just because we started in equilibrium. The prediction is that starting with any, arbitrarily nice, initial data, for *any* time $t > 0$, the solution will be locally Brownian, i.e. (32) will hold, with the same 2 on the right hand side.

We see that the non-linear term needs a kind of infinite renormalization. So it would be more honest to write the equation as[1]

$$\partial_t h = -[\tfrac{1}{2}(\partial_x h)^2 - \infty] + \tfrac{1}{2}\partial_x^2 h + \xi. \tag{33}$$

[1]Before the mathematically inclined reader falls into despair, we should say that we will be completely precise about what we mean by all this later. At this point we are simply providing physical background to the problem.

We have here a problem of scales in the derivation of the process. The nonlinear term is really being computed on a larger scale and is not supposed to be seeing the small scale fluctuations.

1.7. **Cutoff KPZ models.** Now what a stochastic pde is supposed to mean is that we introduce some sort of cutoff, so that the equation makes usual sense, and then remove it to get our solution. For example, we could smooth out the noise a little, with some sort of mollifier, to get ξ_ϵ. If $\xi_\epsilon(t,x)$ were now a smooth function of space and time, we could simply solve

$$\partial_t h_\epsilon = -\tfrac{1}{2}(\partial_x h_\epsilon)^2 + \tfrac{1}{2}\partial_x^2 h_\epsilon + \xi_\epsilon \tag{34}$$

for each realization of the noise separately. Then we try to take a limit as $\epsilon \to 0$ to get our solution. Or we could discretize the equation, for example in space, so that we have

$$\partial_t h(\tfrac{i}{n}) = -Q(h(\tfrac{i+1}{n}), h(\tfrac{i}{n}), h(\tfrac{i-1}{n}), h(\tfrac{i-2}{n})) + n^2(h(\tfrac{i+1}{n}) - 2h(\tfrac{i}{n}) + h(\tfrac{i-1}{n})) + n^{1/2} dB_{\frac{i}{n}} \tag{35}$$

where $Q(h(\frac{i+1}{n}), h(\frac{i}{n}), h(\frac{i-1}{n}), h(\frac{i-2}{n}))$ is some sort of discretization of $\frac{1}{2}(\partial_x h)^2$. Another possibility is that h could be some sort of Markov chain or Markov process which might approximate h.

The last is in some sense the key point. The KPZ equation itself was built to model such systems, so it is crucial that whatever sense we try to make of it, it does model them correctly. Let us look at a few classic examples. All of them will be Markov processes on integer valued height functions h_i on a one dimensional lattice $\mathbb{Z}$.

- **Ballistic aggregation.** At each site one has "arrivals" occuring as a Poisson process with rate one. The arrivals at different sites are independent. When there is an arrival, the update rule is
$$h_i \mapsto \max\{h_{i-1}, h_i + 1, h_{i+1}\}. \tag{36}$$
The arrivals can be thought of as particles which attach to the interface at the highest point at which there is either a particle to the right or left or below it.
- **Eden model.** We grow a finite connected subset A of $\mathbb{Z}^2$ by adding sites in the exterior boundary (i.e. sites in the complement of A which have a nearest neighbour in A). All such sites are added at rate one. To get a growing height function we let $h_i = \min\{j : (i, j+k) \notin A \text{ for all } k = 1, 2, \dots\}$.
- **Restricted solid-on-solid/asymmetric exclusion.** The height profile is restricted to be a nearest neighbour random walk, i.e. $h_{i+1} - h_i \in \{-1, +1\}$. Each pair $(i, i+1)$ has two independent Poisson clocks which ring at rate p and at rate q. When the first clock rings, if the height differences there look like $(1,-1)$ we change it to $(-1,1)$. Otherwise we do nothing. When the second clock rings, if the height differences there look like $(-1,1)$ we change it to $(1,-1)$ and otherwise we do nothing.

The KPZ equation was introduced to model such growth processes, especially the ballistic aggregation. In fact, to this day we know almost nothing about the Eden model or the ballistic aggregation. But the restricted solid on solid model/asymmetric exclusion process turns out to be solvable in a sense. Along with another solvable discretization called PNG (polynuclear growth model) [Joh03], [Joh00], [PS02a] it has been the basis of the recent breakthroughs.

One reason that asymmetric exclusion can be analyzed is that with appropriate boundary conditions its invariant measures are well understood. For example, if we put it on an infinite lattice, and have the flips happen as independent Poisson processes, the invariant measures are two sided random walks, which are discrete versions of Brownian motion. For a discretization like (35) one has to be very careful with the choice of Q if one wants to have a nice invariant measure. For example, the very special discretization

$$Q(h_{i+1}, h_i, h_{i-1}, h_{i-2})) = (h_{i+1} - h_{i-2})(h_{i+1} - h_i - h_{i-1} + h_{i-2}) \tag{37}$$

works, in the sense that it preserves a random walk with Gaussian increments. This discretization was discovered by N. J. Zabusky [NK65] in the context of the Kortweg deVries equation (KdV) which bears many similarities to KPZ.

It is expected that if these cutoff versions of KPZ are not chosen carefully, they may not approximate KPZ. There could be no convergence, or convergence to a different answer. There are very few models in which we can prove any sort of convergence. They will be described in Section 3.12. For (37) we do not know the convergence, though it is expected to be correct (after subtraction of an appropriate diverging constant). For the invariant measures also, we only know them for a few special models. But the relation between having an explicit invariant measure and the convergence is also murky. It is not being used in what proofs we have in a way that is clear.

Another approach to understanding the KPZ equation is to introduce some sort of Wick ordered version of the nonlinearity,

$$:(\partial_x h)^2: \tag{38}$$

which is supposed to reflect what happens to $(\partial_x h)^2$ after the "∞" has been removed. Many attempts in this direction have unfortunately led to non-physical solutions [Cha00][2]

Nevertheless, as we will see, there is a limiting object which in some sense satisfies the KPZ equation. The object is very canonical and we have even been able to obtain exact formulas for various quantities associated with it (see e.g. Theorem 2.3).

Perhaps a more straightforward approach would be to develop a theory for smoothed out noises. Everything would make sense. The problem is that there is some magic in the solution of KPZ and we would miss it this way. It is a classic mathematical situation in which one has to work hard to make sense of a canonical limiting problem, but then has an advantage of some exact solvability.

1.8. **Hopf-Cole solutions.** In the mid 90's L. Bertini and G. Giacomin [BG97], following a suggestion of E. Presutti, proposed that the correct solutions of KPZ could be obtained as follows. The stochastic heat equation (with multiplicative noise) is

$$\partial_t z = \tfrac{1}{2}\partial_x^2 z - z\xi. \tag{39}$$

It has to be interpreted in the Itô sense, in which case it is well posed and for (reasonable) initial data $z_0(x) \geq 0$ with $\int z_0(x)dx > 0$ we have, for later times $t > 0$, $z(t,x) > 0$ for all x (see Theorem 3.10). Bertini and Giacomin proposed that

$$h(t,x) = -\log z(t,x) \tag{40}$$

is the correct solution of KPZ. There are multiple reasons to support this.

(1) If ξ were a nice function, then h would solve (1). This is just the classic Hopf-Cole transformation.

[2] The reason is that their Wick products are based on the Gaussian structure of the forcing white noise ξ. The recent breakthrough work of Martin Hairer [Hai11] uses the Gaussian structure of the Ornstein-Uhlenbeck process obtained by linearizing the KPZ equation, and this leads to the physical solutions.

(2) Let's see what happens when ξ is a white noise. We start with the solution $z(t,x)$ of (39) and smooth it out a bit in space, then look for the equation it satisfies. For the computation we will need to assume we are in equilibrium. Let $\mathcal{G}_\kappa(x) = \frac{1}{\sqrt{2\pi\kappa^2}}\exp\{-x^2/2\kappa^2\}$ and

$$z_\kappa(t,x) = \langle \mathcal{G}_{x,\kappa}, z(t)\rangle = \textstyle\int \mathcal{G}_\kappa(x-y)z(t,y)dy. \tag{41}$$

Define $h_\kappa(t,x) = -\log z_\kappa(t,x)$. Then by Itô's formula,

$$\partial_t h_\kappa + \tfrac12(\partial_x h_\kappa)^2 - \tfrac12\partial_x^2 h_\kappa - \xi = \left\{z_\kappa^{-1}\langle \mathcal{G}_{x,\kappa} z, \xi\rangle - \xi\right\} + \tfrac12 z_\kappa^{-2}\langle \mathcal{G}_{x,\kappa}^2, z^2\rangle. \tag{42}$$

For the first term, we can compute $E[(\int\int \varphi(t,x)\{z_\kappa^{-1}\langle \mathcal{G}_{x,\kappa}z,\xi\rangle - \xi\}dxdt)^2]$ for a smooth function φ of compact support by Itô isometry to get

$$\textstyle\int\int E[(\int \frac{\varphi(t,y)\mathcal{G}_\kappa(y-x)dy}{\int \mathcal{G}_\kappa(y-y')z(t,y')dy'} z(t,x) - \varphi(t,x))^2]dxdt, \tag{43}$$

which vanishes as $\kappa \searrow 0$ by the continuity of $z(t,x)$.

We now compute the last term. Define $\mathcal{J}_\kappa(x) = 2\kappa\sqrt{\pi}\mathcal{G}_\kappa^2(x)$ so that $\mathcal{J}_\kappa$, $\kappa > 0$ is a new approximate identity. The last term is

$$\tfrac14\kappa^{-1}\pi^{-1/2}\textstyle\int \mathcal{J}_\kappa(x-y)e^{2(h(y)-h(x))}dy\ (\int \mathcal{G}_\kappa(x-y)e^{h(y)-h(x)}dy)^{-2}. \tag{44}$$

Because we are in equilibrium the $h(y)-h(x)$ are Brownian increments and we can make what is essentially a quadratic variation computation to get

$$\tfrac12 z_\kappa^{-2}\langle \mathcal{G}_{x,\kappa}^2, z^2\rangle \sim \tfrac14\kappa^{-1}\pi^{-1/2}. \tag{45}$$

The precise constant $\frac12\pi^{-1/2}$ just comes from the L^2 norm of the Gaussian smoothing. We could smooth with something else and get another constant.

So

$$\partial_t h_\kappa = -\tfrac12[(\partial_x h_\kappa)^2 - \tfrac12\kappa^{-1}\pi^{-1/2}] + \tfrac12\partial_x^2 h_\kappa + \xi + o(1). \tag{46}$$

This gives our first precise form of (33).

(3) Suppose that instead of smoothing out z as above, we smooth out the white noise (in space),

$$\xi^\kappa(t,x) = \textstyle\int \mathcal{G}_\kappa(x-y)\xi(t,y)dy.$$

Since the operation is linear $\xi^\kappa(t,x)$ is still Gaussian. It's mean is still zero and it has covariance

$$E[\xi^\kappa(t,x)\xi^\kappa(s,y)] = C_\kappa(x-y)\delta(t-s)$$

where

$$C_\kappa(x-y) = \int \mathcal{G}_\kappa(x-u)\mathcal{G}_\kappa(y-u)du.$$

In particular, we have again

$$C_\kappa(0) = \tfrac12\kappa^{-1}\pi^{-1/2}. \tag{47}$$

Let $z^\kappa(t,x)$ be the solution of the stochastic heat equation with the smoothed noise

$$\partial_t z^\kappa = \partial_x^2 z^\kappa - z^\kappa\xi^\kappa, \qquad t>0, \quad x\in\mathbb{R}. \tag{48}$$

It is not difficult to show that $z^\kappa \to z$ uniformly on compact sets, and because $z(t,x) > 0$ for $t > 0$, we can define

$$h^\kappa(t,x) = -\log z^\kappa(t,x). \tag{49}$$

and $h^\kappa(t,x)$ converge to $h(t,x) = -\log z(t,x)$. By Itô's formula, we have

$$\partial_t h^\kappa = -\tfrac{1}{2}[(\partial_x h^\kappa)^2 - C_\kappa(0)] + \tfrac{1}{2}\partial_x^2 h^\kappa + \xi^\kappa. \tag{50}$$

Compare to (46).

(4) The Hopf-Cole solution is the one obtained by approximating KPZ by the free energy of directed random polymers and by the height function of asymmetric exclusion [BG97] (see Section 3.12). This weakly asymmetric limit is expected to hold for a large class of systems with an adjustable asymmetry but proofs are not available at this time (see [GJ10] for recent partial results).

(5) The Hopf-Cole solution has the conjectured scaling exponents [BQS11]. One can also obtain some of the conjectured asymptotic fluctuations [ACQ11, SS10a, SH10].

The evidence for the Hopf-Cole solutions is now overwhelming. Whatever the physicists mean by KPZ, it is them. The problem of proving well-posedness for (1)-(3) is now seen to be of a very different nature from a problem like well-posedness for the 3-d incompressible Navier-Stokes equations. In the present case, we know that the solution is the Hopf-Cole solution. The problem is to find an appropriate definition of (1)-(3) which fits that solution, and to prove the corresponding uniqueness. As these notes were being produced, a solution to this problem has been announced by Martin Hairer [Hai11][3]

Note that in higher dimensions one also has the formal Hopf-Cole formula linking

$$\partial_t h = -\tfrac{1}{2}|\nabla h|^2 + \tfrac{1}{2}\Delta h + \xi \tag{51}$$

with

$$\partial_t z = \tfrac{1}{2}\Delta z - z\xi \tag{52}$$

via

$$h(t,x) = -\log z(t,x). \tag{53}$$

Unfortunately, it can be checked that (52) does not have function valued solutions except in one dimension. So (53) does not make sense.

Discrete versions do hold. In particular, the partition functions of directed polymers in $d+1$ dimensions satisfy discrete versions of (52), and their logarithm (the free energy) satisfies a discrete version of (51). We now describe them.

1.9. **Directed polymers in a random environment.** The random environment is a collection $\xi(i,j)$ of independent identically distributed real random variables placed on the sites i,j of $\mathbb{Z}_+ \times \mathbb{Z}^d$. Given a realization $\xi(i,j)$ of the environment, the energy of an n-step nearest neighbour walk $\mathbf{x} = (x_1, \dots, x_n)$ is

$$H_n(\mathbf{x}) = \sum_{i=1}^n \xi(i, x_i).$$

Nearest neighbour walk just means a sequence of integers x_i, $i = 0, 1, \dots, n$, with x_{i+1} an adjacent lattice site to x_i. The polymer measure on such walks starting at x at time 0 and ending at y at time n is then defined by

$$P_{x,n,y}(\mathbf{x}) = \frac{1}{Z(x,n,y)} e^{-\beta H_n(\mathbf{x})}\, P(\mathbf{x}),$$

[3] Technically, the results of [Hai11] are for KPZ on $[0, 2\pi)$ with periodic boundary conditions, and extending them to $\mathbb{R}$ remains an open problem.

where $\beta > 0$ is a parameter called the inverse temperature, P is the uniform probability measure on such walks, and $Z(x, n, y)$ is the partition function

$$Z(x, n, y) = \sum_{\mathbf{x}} e^{-\beta H_n^\omega(\mathbf{x})} P(\mathbf{x})$$

This is called the point-to-point polymer. If we do not specify the end point, we call it the point-to-line polymer, and we call the partition function $Z(n)$. We will usually suppress the dependence on ξ and β except where it may cause confusion, in which case we write things like $Z_\beta(n)$. For each realization of the environment we have a probability measure on random walk paths that prefers to travel through areas of low energy. Since the environment is thought of as random, we have a random probability measure on random walk paths.

They were introduced by [HH85] as a model of domain walls in Ising systems and arise as competition interfaces in multi-species growth [OH07]. There is a war between entropy (the number of such walks) and the energy. At $\beta = 0$ the polymer measure is of course just simple random walk, hence the walk is entropy dominated and exhibits diffusive behaviour. For β large, the polymer measure concentrates on special low energy paths which are no longer diffusive. Entropy domination is called *weak disorder*, and energy domination is called *strong disorder*. The precise separation between these two regimes is defined mathematically in terms of the positivity of the limit of the martingale $e^{-n\lambda(\beta)}Z(n)$ for the point-to-line partition function, as $n \to \infty$, where $\lambda(\beta) = E[e^{-\beta\omega}]$. The weak disorder regime consists of β for which

$$\lim_{n\to\infty} e^{-n\lambda(\beta)} Z_\beta(n) > 0,$$

whereas if the limit is zero then β is said to be in the strong disorder regime. For $d \geq 3$, it was shown early on [IS88, Bol89] that weak disorder holds for small β. Later, Comets and Yoshida [CY06] showed that in every dimension there is a critical value β_c such that weak disorder holds for $0 \leq \beta < \beta_c$ and strong disorder for $\beta > \beta_c$. In addition, for $d = 1$ and 2 they prove that $\beta_c = 0$. Understanding the polymer behaviour in the strong disorder regime is the main goal. The paths are superdiffusive with (point-to-line case)

$$|x(n)| \sim n^\zeta$$

for n large, with $\zeta > 1/2$. For $d = 1$ the long-standing conjecture is $\zeta = 2/3$. For a long time we there were only non-sharp upper and lower bounds for special models [CY05, Mej04, Pet00, Wüt98a, Wüt98b]. Note that the picture is very different from simple random walk where the polymer endpoint is roughly uniformly distributed on an interval of size $\sqrt{n}$. For each realization of the random environment, the polymer is *localized* near a point of distance about $n^{2/3}$ from the starting point. This point of course depends on the random environment. So the randomness in the polymer endpoint is basically a function of the random environment, and not the randomness of the random walk paths. Carmona and Hu [CH02] and Comets et al [CSY03] showed that there is a constant $c_0 = c_0(\beta) > 0$ such that for the point-to-line

$$\overline{\lim_{n\to\infty}} \sup_{x\in\mathbb{Z}} P^\omega_{n,\beta}(x(n) = x) \geq c_0.$$

has probability one, in stark contrast to the simple random walk case where the supremum decays like $n^{-1/2}$.

Strong disorder also can be seen in the large time behaviour of the partition function. There is a strict inequality

$$\rho(\beta) := \lim_{n\to\infty} \tfrac{1}{n}\log Z_\beta(n) = \lim_{n\to\infty} \tfrac{1}{n} E \log Z_\beta(n) < \lim_{n\to\infty} \tfrac{1}{n}\log E Z_\beta(n) := \lambda(\beta) \tag{54}$$

between the quenched and annealed free energies (the second equality is by a subadditivity argument and some concentration estimates, see for example [CH02, CSY03]). $\leq$ is obvious. The fact that it is strict was proved in $d = 1$ by Comets et al [CSY04] (see also [Lac09]). From (54) the leading term behavior of the log of the partition function is $\rho(\beta)n$.

The randomness is conjectured to appear through the random second order term

$$\log Z_\beta(n) \sim \rho(\beta)n + c(\beta)n^\chi X. \tag{55}$$

ζ and χ are supposed to satisfy the KPZ relation,

$$\chi = 2\zeta - 1. \tag{56}$$

Partial rigorous results were obtained recently by Chatterjee [Cha11] (see also [AD11]).

In the following we will only be interested in $d = 1$. The conjecture is $\chi = 1/3$ and $\zeta = 2/3$. Results on $\zeta = 2/3$ have recently been obtained in special models [BQS11],[Sep09].

1.10. **LPP, PNG, TASEP and F_{GUE}.** Now we turn to several models which are in some sense solvable and which the source of the preceding conjectures about the asymptotic fluctuations. Although these are truly mathematical results, we do not describe the mathematical details, but use the results to inspire the physical picture for KPZ.

Let $\xi(i,j)$, $(i,j) \in \mathbb{Z}_+^2$, be independent and identically distributed random variables. An *up/right path* $\mathbf{x}$ from $(1,1)$ to (M,N) is a sequence $(1,1) = (i_1,j_1),(i_2,j_2),\dots,(i_m,j_m) = (M,N)$, $m = M+N-1$, such that either $i_{r+1} - i_r = 1$ and $j_{r+1} = j_r$, or $i_{r+1} = i_r$ and $j_{r+1} - j_r = 1$. Set

$$G(M,N) = \max_{\mathbf{x}} \sum_{(i,j)\in\pi} \xi(i,j), \tag{57}$$

where the maximum is taken over all up/right paths $\mathbf{x}$ from $(1,1)$ to (M,N).

One can see that last passage percolation is a kind of zero temperature $\beta \to \infty$ limit of the directed polymer in random environment, with $G(N,M)$ replacing the free energy. In fact it satisfies

$$G(M,N) = \max(G(M-1,N), G(M,N-1)) + \xi(M,N). \tag{58}$$

This is another discrete version of the KPZ equation (not in the sense that it scales to the KPZ equation, but in the sense that it is a discrete model with the same structure, and which is in the KPZ class.) Note that if we had been careful, we could even have defined the discrete polymers so that the last passage percolation is exactly the zero temperature limit.

If $\xi(i,j)$ have the geometric distribution,

$$P(\xi(i,j) = k) = (1-q)q^k, \qquad k \in \{0,1,2,\dots\}$$

this last passage percolation model can be seen as a discrete version of the polynuclear growth model (PNG) [KS92], [PS00]. Let $h(x,t) \in \mathbb{N}$ denote the height above $x \in \mathbb{Z}$ at time $t \in \mathbb{N}$. The model is defined by the discrete KPZ equation

$$h(x,t+1) = \max(h(x-1,t), h(x,t), h(x+1,t)) + \tilde{\xi}(x,t), \tag{59}$$

where $\xi(x,t)$ are independent random variables for $t \in \mathbb{Z}_+$ and $x \in \mathbb{Z}$ which vanish whenever $x-t$ is even, and have the geometric distribution when $x-t$ is odd.

The connection between last passage percolation and this discrete PNG model is that if $\xi(i,j) = \tilde{\xi}(i-j, i+j-1)$, then

$$G(i,j) = h(i-j, i+j-1). \tag{60}$$

In this case we have a first example of an exact formula. If $M \geq N$,

$$P(G(M,N) \leq s) = \det(I - K_N)_{L^2((s,\infty))}$$

where

$$K_N(x,y) = \frac{\kappa_{N-1}}{\kappa_N} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y}(w(x)w(y))^{1/2}. \tag{61}$$

$$w(x) = \binom{M-N+x}{x}$$

$p_N(x)$ are the Meixner polynomials, i.e. the normalized orthogonal polynomials $p_n(x) = \kappa_n x^n + \dots$ with respect to the measure $w(x)dx$.

This illustrates the connection between growth models and random matrices: By taking appropriate asymptotic limits of this formula one obtains

$$G(N,N) \sim c_1 N + c_2 N^{1/3}\zeta \tag{62}$$

where ζ has the GUE Tracy-Widom distribution F_{GUE} which we now describe.

Consider the largest eigenvalue $\lambda_N^{\max}$ of a matrix from the Gaussian unitary ensemble (GUE), i.e. a Hermetian $N \times N$ matrix $a_{ij} = \overline{a_{ji}}$, $i,j = 1,\dots,N$ such that for $i < j$, a_{ij} is distributed as $\mathscr{N}(0,\sqrt{N}/2) + i\mathscr{N}(0,\sqrt{N}/2)$ and on the diagonal $i=j$, a_{ii} is distributed as $\mathscr{N}(0,\sqrt{N})$ where $\mathscr{N}(a,b)$ means Gaussian (=normal) mean 0 and variance b. The a_{ii} and the real and imaginary parts of the a_{ij} are all independent. The $\sqrt{N}$ is an arbitrary normalization which makes the analogy with growth models more transparent. Alternatively, the probability measure on the space of Hermitian matrices is

$$Z_N^{-1} e^{-\frac{1}{2N}\mathrm{Tr}A^2} \prod_{i=1}^{N} da_{ii} \prod_{i<j=1}^{N} d\mathrm{Re}a_{ij} d\mathrm{Im}a_{ij}. \tag{63}$$

By the Wigner semicircle law, the spectrum has approximately a semicircle density on $[-2N, 2N]$ and

$$\lambda_N^{\max} \sim 2N + N^{1/3}\zeta \tag{64}$$

where ζ has the GUE Tracy-Widom distribution,

$$P(\zeta \leq s) = \det(I - P_s K_{\mathrm{Ai}} P_s) \tag{65}$$

where $P_s f(x) = \mathbf{1}_{x \leq s} f(x)$ is the projection onto $L^2(s,\infty)$ and the *Airy kernel*

$$K_{\mathrm{Ai}}(x,y) = \int_{-\infty}^{0} \mathrm{Ai}(x-\lambda)\mathrm{Ai}(y-\lambda)d\lambda. \tag{66}$$

is the projection onto the negative eigenspace of the Airy operator

$$H = -\partial_x^2 + x. \tag{67}$$

Here, and in everything that follows, the determinant means the Fredholm determinant in the Hilbert space $L^2(\mathbb{R})$.

In the totally asymmetric case $p = 0, q = 1$ [Joh00] and much later in the asymmetric case $p > 0, p + q = 1$ [TW09a] the fluctuations of asymmetric exclusion were also shown to be asymptotically F_{GUE}: *Let $q - p \in (0, 1]$ and start with $h(0, x) = |x|$, then*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{h(\frac{t}{q-p}, 0) - \frac{t}{2}}{2^{-1/3}t^{1/3}} \geq -s \right) = F_{\mathrm{GUE}}(s).$$

2. F_{GUE} conjecture for directed polymers and KPZ

Based on the results of the last section the key conjecture for the point-to-point free energy of discrete directed polymer models is that

$$\log Z_\beta(n) \sim \rho(\beta)n + c(\beta)n^\chi X. \tag{68}$$

where $X \sim F_{\mathrm{GUE}}$. In the point-to-line case, the GOE Tracy-Widom distribution. At finite temperature ($\beta > 0$) the only results are for the continuum random polymer and the corollaries for intermediate disorder.

The point-to-point partition function $Z^\omega_\beta(x, n, y)$ is very analogous to the solution of the stochastic heat equation. It satisfies a forward equation

$$Z(x, j + 1, y) = \frac{1}{2} e^{-\beta\xi(j+1,y)} \left[Z(x, j, y + 1) + Z(x, j, y - 1)\right] \tag{69}$$

with the initial data $Z(x, 0, y) = \delta(x = y)$. This can be recognized as a discrete version of the stochastic heat equation, which we know is the Hopf-Cole transformation of the KPZ equation (1). Hence the conjecture is that if $h(t, x)$ is the *narrow wedge solution of KPZ*, meaning (40) where $Z(0, x) = \delta_0(x)$, then

$$h(t, x) \sim \tfrac{x^2}{2t} + \tfrac{t}{24} + 2^{-1/3}t^{1/3}X \tag{70}$$

where X has the *GUE* Tracy-Widom distribution.

In Section 3.13 we will show that they converge to it under an appropriate scaling.

The Dyson Brownian motion is a stationary time dependent version of the ensemble which has the GUE as its distribution at every time. The matrices evolve according to the Ornstein-Uhlenbeck process given by the Langevin equation,

$$dA = -\tfrac{1}{2N} A dt + dB \tag{71}$$

where $B_{ii}(t)$, $i = 1, \dots, N$, $\mathrm{Re}B_{ij}(t)$, $\mathrm{Im}B_{ij}$, $i < j = 1, \dots, N$ are independent Brownian motions, the first with diffusion coefficient 1 and the latter two with diffusion coefficients 1/2. The largest eigenvalue $\lambda^{\max}_N(t)$ of $A(t)$ is now a function of t and

$$\lambda^{\max}_N(t) \sim 2N + N^{1/3}\mathcal{A}_2(t) \tag{72}$$

where the Airy$_2$ process $\mathcal{A}_2$ is defined through its finite-dimensional distributions, which are given by a determinantal formula: given $x_0, \dots, x_n \in \mathbb{R}$ and $t_0 < \cdots < t_n$ in $\mathbb{R}$,

$$P(\mathcal{A}_2(t_0) \leq x_0, \dots, \mathcal{A}_2(t_n) \leq x_n) = \det(I - \mathrm{f}^{1/2}K_{\mathrm{ext}}\mathrm{f}^{1/2})_{L^2(\{t_0,\dots,t_n\}\times\mathbb{R})}, \tag{73}$$

where we have counting measure on $\{t_0, \dots, t_n\}$ and Lebesgue measure on $\mathbb{R}$, f is defined on $\{t_0, \dots, t_n\} \times \mathbb{R}$ by $\mathrm{f}(t_j, x) = \mathbf{1}_{x\in(x_j,\infty)}$, and the *extended Airy kernel* [PS02b, FNH99, Mac94] is defined by

$$K_{\mathrm{ext}}(t, \xi; t', \xi') = \begin{cases} \int_0^\infty d\lambda\, e^{-\lambda(t-t')}\mathrm{Ai}(\xi + \lambda)\mathrm{Ai}(\xi' + \lambda), & \text{if } t \geq t' \\ \int_{-\infty}^0 d\lambda\, e^{-\lambda(t-t')}\mathrm{Ai}(\xi + \lambda)\mathrm{Ai}(\xi' + \lambda), & \text{if } t < t', \end{cases}$$

Here we have only scratched the surface of the solvable models. For nice review articles the reader is directed to [Fer10], [FS10], [Joh06], especially for the relation between last passage percolation and the RSK (Robinson-Schensted-Knuth) correspondence. A very recent article [COSZ11] discusses many such exact formulas in the context of the tropical version of RSK (see also [BC11] for very general results about solvability in this context.)

2.1. $F_{\rm GOE}$ **and Airy$_1$.** Now we consider the largest eigenvalue $\lambda_N^{\max}$ of a matrix from the Gaussian orthogonal ensemble (GOE). The probability measure on the space of real symmetric matrices is

$$Z_N^{-1} e^{-\frac{1}{2N}\mathrm{Tr}A^2} \prod_{i\le j=1}^{N} da_{ij}. \tag{74}$$

As before we have

$$\lambda_N^{\max} \sim 2N + N^{1/3}\zeta \tag{75}$$

but now ζ has the Tracy-Widom GOE distribution which can be written [FS05]

$$P(\zeta \le s) = \det(I - P_s B P_s) \tag{76}$$

where

$$B(x,y) = \mathrm{Ai}(x+y).$$

The Airy$_1$ process is another stationary process, whose one-point distribution is now $F_{\rm GOE}$. It is defined through its finite-dimensional distributions, given by a determinantal formula: for $x_1,\dots,x_n\in\mathbb{R}$ and $t_1<\dots<t_n$ in $\mathbb{R}$,

$$P(\mathcal{A}_1(t_1)\le x_1,\dots,\mathcal{A}_1(t_n)\le x_n) = \det(I - \mathrm{f}^{1/2}K_1^{\rm ext}\mathrm{f}^{1/2})_{L^2(\{t_1,\dots,t_n\}\times\mathbb{R})}, \tag{77}$$

where we have counting measure on $\{t_1,\dots,t_n\}$ and Lebesgue measure on $\mathbb{R}$, f is defined on $\{t_1,\dots,t_n\}\times\mathbb{R}$ by $\mathrm{f}(t_j,x)=\mathbf{1}_{x\in(x_j,\infty)}$ and

$$K_1^{\rm ext}(t,x;t',x') = -\frac{1}{\sqrt{4\pi(t'-t)}}\exp\left(-\frac{(x'-x)^2}{4(t'-t)}\right)\mathbf{1}_{t'>t} \tag{78}$$

$$+\mathrm{Ai}(x+x'+(t'-t)^2)\exp\left((t'-t)(x+x')+\frac{2}{3}(t'-t)^3\right). \tag{79}$$

The finite-dimensional distributions of the Airy$_1$ process are also given by the following formula: for $x_1,\dots,x_n\in\mathbb{R}$ and $t_1<\dots<t_n$ in $\mathbb{R}$,

$$P(\mathcal{A}_1(t_1)\le x_1,\dots,\mathcal{A}_1(t_n)\le x_n) \tag{80}$$

$$= \det\big(I - B_0 + \bar{P}_{x_1}e^{-(t_1-t_2)\Delta}\bar{P}_{x_2}e^{-(t_2-t_1)\Delta}\cdots\bar{P}_{x_n}e^{-(t_n-t_1)\Delta}B_0\big)_{L^2(\mathbb{R})}. \tag{81}$$

Note however, that it is not true that the Airy$_1$ process is the limit of the largest eigenvalue process in the matrix valued diffusion for GOE [BFP08] (and it is an open problem what is.) Airy$_1$ arises in growth models starting from flat initial data and point-to-line random polymers [Sas05],[BFPS07].

2.2. **Equilibrium.** The equilibrium analogue of the Airy_1 and Airy_2 processes we call $\text{Airy}_{\text{stat}}$, or $\mathcal{A}_{\text{st}}(t)$. It has finite dimensional distributions [BFP10]

$$P(\mathcal{A}_{\text{st}}(t_1) \le x_1, \dots, \mathcal{A}_{\text{st}}(t_n) \le x_n) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left(g_n(t,x) \det \left(I - \mathrm{f}^{1/2} \widehat{K}_{\text{Ai}} \mathrm{f}^{1/2} \right)_{L^2(\{t_1,\dots,t_n\}\times\mathbb{R})} \right) \tag{82}$$

where

$$\widehat{K}_{\text{Ai}}((i,x),(j,y)) = \begin{cases} \displaystyle\int_0^\infty d\lambda \text{Ai}(x+\lambda+t_i^2)\text{Ai}(y+\lambda+t_j^2)e^{-\lambda(t_j-t_i)}, & \text{if } t_i \le t_j, \\ \displaystyle -\int_{-\infty}^0 d\lambda \text{Ai}(x+\lambda+t_i^2)\text{Ai}(y+\lambda+t_j^2)e^{-\lambda(t_j-t_i)}, & \text{if } t_i > t_j. \end{cases} \tag{83}$$

The function $g_n(t,x)$ is defined by

$$g_n(t,x) = \mathcal{R} + \sum_{i=1}^{n}\sum_{j=1}^{n} \int_{x_i}^\infty dx' \int_{x_j}^\infty dy' \, \Psi_j(y')\rho_{j,i}(y',x')\Phi_i(x'), \tag{84}$$

where

$$\rho := (I - \mathrm{f}^{1/2}\widehat{K}_{\text{Ai}}\mathrm{f}^{1/2})^{-1}, \qquad \rho_{j,i}(y,x) := \rho((j,y),(i,x)), \tag{85}$$

and $\Phi((i,x)) := \Phi_i(x)$, $\Psi((j,y)) = \Psi_j(y)$. The functions $\mathcal{R}$, Φ, and Ψ are defined by

$$\begin{aligned} \mathcal{R} &= x_1 + e^{-\frac{2}{3}t_1^3} \int_{x_1}^\infty dx \int_0^\infty dy \, \text{Ai}(x+y+t_1^2)e^{-t_1(x+y)}, \\ \Psi_j(y) &= e^{\frac{2}{3}t_j^3 + t_j y} - \int_0^\infty dx \, \text{Ai}(x+y+t_j^2)e^{-t_j x}, \\ \Phi_i(x) &= e^{-\frac{2}{3}t_1^3} \int_0^\infty d\lambda \int_{x_1}^\infty dy \, e^{-\lambda(t_1-t_i)} e^{-t_1 y} \text{Ai}(x+t_i^2+\lambda)\text{Ai}(y+t_1^2+\lambda) \\ &\quad + \mathbf{1}_{[i\ge 2]} \frac{e^{-\frac{2}{3}t_i^3 - t_i x}}{\sqrt{4\pi(t_i-t_1)}} \int_{-\infty}^{x_1-x} dy \, e^{-\frac{y^2}{4(t_i-t_1)}} - \int_0^\infty dy \, \text{Ai}(y+x+t_i^2)e^{t_i y}. \end{aligned} \tag{86}$$

for $i,j = 1,2,\dots,n$. The marginal $\mathcal{A}_{\text{st}}(0)$ has the Baik-Rains distribution, sometimes called F_0) [BR00], [FS06].

2.3. **Predicted fluctuations for KPZ.** Extrapolating from the exact results for TASEP and PNG, one has the following predictions for fluctuations for models in the KPZ universality class. These depend on the initial conditions. Recalling that the Hopf-Cole solution $h(t,x) = -\log z(t,x)$ where $z(t,x)$ is the solution of the stochastic heat equation,

$$\partial_t z = \tfrac{1}{2}\partial_x^2 z + \xi z \tag{87}$$

we can state the conjectures in terms of the special initial conditions for $z(t,x)$. The initial conditions are singled out by certain scale invariance. There are three most important ones: 1. *Curved*, corresponding to initial data $z(0,x) = \delta_0$; 2. *Flat*, corresponding to initial data $z(0,x) = 1$; 3. *Equilibrium*, corresponding to $z(0,x) = e^{B(x)}$ where $B(x)$ is a two-sided Brownian motion.

The conjecture is that in each case the spatial fluctuations at time t are of size $t^{1/3}$ and live on a scale $t^{2/3}$, with approximating processes Airy_2 in the curved case, Airy_1 in the flat case, and $\text{Airy}_{\text{stat}}$ in the equilibrium case. One always has to subtract appropriate functions of t and possibly x.

The precise conjecture in the curved case is as follows. Let $z(t,x)$ be the solution of the stochastic heat equation (87) with initial data $z(0,x)=\delta_0$ and let $h(t,x)=-\log z(t,x)$ Then, as processes, as $t\to\infty$,

$$2^{1/3}t^{-1/3}(h(t,t^{2/3}x)+\tfrac{x^2}{2t}+\log\sqrt{2\pi t}+\tfrac{t}{24})\to\mathcal{A}_2(x). \tag{88}$$

2.4. **Statement of main mathematical results.** We have the following results in the direction of the previous conjectures.

Theorem 2.1 (Equilibrium. [BQS11])**.** *Let $z(t,x)$ be the solution of the stochastic heat equation (87) with initial data $z(0,x)=e^{B(x)}$. Let $h(t,x)=-\log z(t,x)$ be the corresponding solution of KPZ. There are constants $0<c_1<c_2<\infty$ such that*

$$c_1t^{2/3}\le \mathrm{Var}(h(t,0))\le c_2t^{2/3}. \tag{89}$$

Theorem 2.2 (Curved. [ACQ11], [SS10b])**.** *Let $z(t,x)$ be the solution of the stochastic heat equation (87) with initial data $z(0,x)=\delta_0(x)$. Then the one dimensional distributions of the left hand side of (88) converge to those of the right hand.*

The latter theorem is a relatively easy computation once one has following very unexpected exact formula:

Theorem 2.3 (Curved. [ACQ11], [SS10b])**.** *Let $z(t,x)$ be as in the previous theorem and let $h(t,x)=-\log z(t,x)$ and define*

$$F_t(s)=P(h(t,x)+\tfrac{x^2}{2t}+\log\sqrt{2\pi t}+\tfrac{t}{24}\ge -s). \tag{90}$$

$F_t(s)$ does not depend on x and is given by the crossover formula

$$F_t(s)=\int_C\frac{d\mu}{\mu}e^{-\mu}\det(I-K_{\sigma_{t,\mu}})_{L^2(\kappa_t^{-1}s,\infty)} \tag{91}$$

where $\kappa_t=2^{-1/3}t^{1/3}$, C is a contour positively oriented and going from $+\infty+\epsilon i$ around $\mathbb{R}^+$ to $+\infty-i\epsilon$, and K_σ is an operator given by its integral kernel

$$K_\sigma(x,y)=\int_{-\infty}^{\infty}\sigma(\tau)\mathrm{Ai}(x+\tau)\mathrm{Ai}(y+\tau)d\tau,\qquad \textit{and}\qquad \sigma_{t,\mu}(\tau)=\frac{\mu}{\mu-e^{-\kappa_t\tau}}. \tag{92}$$

The last formula was derived independently by the two groups using the same method. However [ACQ11] provided a complete proof, while at certain points in the derivation [SS10b] proceed based on physical reasoning.

Both Theorems 2.1 and 2.3 are based on approximating KPZ by the weakly asymmetric simple exclusion process (see Section 3.16).

Theorem 2.1 proceeds via formulas which relate the variance of the height function to the variance of a second class particle in the exclusion processes. These are then studied using coupling arguments.

Theorem 2.3 proceeds by a fine steepest descent analysis of the recently discovered exact formulas of Tracy and Widom [TW08b, TW08a, TW09a] for the asymmetric exclusion process starting with the configuration $\{\dots,0,0,0,1,1,1,\dots\}$ (the *corner growth model*).

Theorem 2.4 (Tracy-Widom ASEP formula [TW09a])**.** *Consider the corner growth model with wedge initial conditions and with $q>p$ such that $q+p=1$. Let $\gamma=q-p$ and $\tau=p/q$. For $m=\lfloor\frac12(s+x)\rfloor$, $t\ge0$ and $x\in\mathbb{Z}$*

$$P(h_\gamma(t,x)\ge s)=\int_{S_{\tau^+}}\frac{d\mu}{\mu}\prod_{k=0}^{\infty}(1-\mu\tau^k)\det(I+\mu J_{t,m,x,\mu})_{L^2(\Gamma_\eta)} \tag{93}$$

where S_{τ^+} is a positively oriented circle centered at zero of radius strictly between τ and 1, and where the kernel of the determinant is given by

$$J_{t,m,x,\mu}(\eta,\eta') = \int_{\Gamma_\zeta} \exp\{\Psi_{t,m,x}(\zeta) - \Psi_{t,m,x}(\eta')\} \frac{f(\mu,\zeta/\eta')}{\eta'(\zeta-\eta)} d\zeta \tag{94}$$

where η and η' are on Γ_η, a circle centered at zero of radius strictly between τ and 1, and the ζ integral is on Γ_ζ, a circle centered at zero of radius strictly between 1 and τ^{-1}, and where

$$\begin{aligned} f(\mu,z) &= \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1-\tau^k\mu} z^k \\ \Psi_{t,m,x}(\zeta) &= \Lambda_{t,m,x}(\zeta) - \Lambda_{t,m,x}(\xi) \\ \Lambda_{t,m,x}(\zeta) &= -x\log(1-\zeta) + \frac{t\zeta}{1-\zeta} + m\log\zeta \end{aligned} \tag{95}$$

ASEP is described in Section 3.15. The argument is sketched in a complementary review article [Cor11]. In Section 3.17 we will sketch the similar argument for the step Bernoulli case, which we now describe.

2.5. **Transitional initial conditions.** Besides the three basic initial conditions, curved, flat and equilibrium, there are three more special initial conditions which basically have one of them on one side of the origin, and another on the other side. The asymptotic fluctuations will be governed by transitional Airy processes, $\mathrm{Airy}_{2\to 1}$ [BFS08], $\mathrm{Airy}_{2\to stat}$ [IS04], $\mathrm{Airy}_{1\to stat}$ [BFS09], which look like one of the Airy processes as $x\to-\infty$ and the other as $x\to\infty$. Their finite dimensional distributions are also given by Fredholm determinants.

We also have a crossover version of these facts, which we now describe.

2.6. **KPZ with half Browian initial data.** Let $B(x)$ be a Brownian motion with diffusion coefficient 2 and let $z(t,x)$ be the solution of the stochastic heat equation with

$$\begin{aligned} &\partial_t z = \tfrac12 \partial_x^2 z + \xi z, \qquad t>0,\ x\in\mathbb{R}, \\ &z(0,x) = e^{-B(x)}\mathbf{1}(x\ge 0). \end{aligned} \tag{96}$$

$h(t,x) = -\log z(t,x)$ is called KPZ with half Brownian initial data. From the previous section the following conjecture should not be surprising,

$$\frac{h(t,2^{1/3}t^{2/3}x)-\frac{t}{24}}{t^{1/3}} - 2^{-1/3}x^2 \overset{t\to\infty}{\longrightarrow} \mathrm{Airy}_{2\to stat}(x). \tag{97}$$

We can prove this in the sense of one dimensional marginals, by taking the $t\to\infty$ limit in another surprising exact formula.

Theorem 2.5 ([CQ11a])**.** *Let $h(t,x)$ be the Hopf-Cole solution of KPZ starting from half-Brownian initial data. Let*

$$F_{t,x}^{\mathrm{edge}}(s) := P\left(\frac{h(t,2^{1/3}t^{2/3}x)-\frac{t}{24}}{t^{1/3}} \ge 2^{-1/3}x^2 - s\right). \tag{98}$$

Then

$$F_{t,x}^{\mathrm{edge}}(s) = \int_{\tilde{\mathcal{C}}} e^{-\tilde\mu} \frac{d\tilde\mu}{\tilde\mu} \det(I - K_s^{\mathrm{edge}})_{L^2(\tilde\Gamma_\eta)}.$$

The contour $\tilde{\mathcal{C}}$ is given as

$$\tilde{\mathcal{C}} = \{e^{i\theta}\}_{\pi/2 \leq \theta \leq 3\pi/2} \cup \{x \pm i\}_{x>0}$$

The contours $\tilde{\Gamma}_\eta$, $\tilde{\Gamma}_\zeta$ are given as

$$\begin{aligned} \tilde{\Gamma}_\eta &= \{2^{-7/3} + ir : r \in (-\infty, -1) \cup (1, \infty)\} \cup \tilde{\Gamma}_\eta^d && (99) \\ \tilde{\Gamma}_\zeta &= \{-2^{-7/3} + ir : r \in (-\infty, -1) \cup (1, \infty)\} \cup \tilde{\Gamma}_\eta^d, && (100) \end{aligned}$$

where $\tilde{\Gamma}_\zeta^d$ is a dimple which goes to the right of $xt^{-1/3}$ and joins with the rest of the contour, and where $\tilde{\Gamma}_\eta^d$ is the same contour just shifted to the right by distance $2^{-4/3}$. The kernel K_s^{edge} acts on the function space $L^2(\tilde{\Gamma}_\eta)$ through its kernel:

$$K_s^{\mathrm{edge}}(\tilde{\eta}, \tilde{\eta}') = \tag{101}$$

$$\int_{\tilde{\Gamma}_\zeta} \exp\{-\frac{t}{3}(\tilde{\zeta}^3 - \tilde{\eta}'^3) + st^{1/3}(\tilde{\zeta} - \tilde{\eta}')\} \frac{\pi 2^{1/3}(-\tilde{\mu})^{-2^{1/3}(\tilde{\zeta}-\tilde{\eta}')}}{\sin(\pi 2^{1/3}(\tilde{\zeta} - \tilde{\eta}'))} \frac{\Gamma(2^{1/3}\tilde{\zeta} - 2^{1/3}xt^{-1/3})}{\Gamma(2^{1/3}\tilde{\eta}' - 2^{1/3}xt^{-1/3})} \frac{d\tilde{\zeta}}{\tilde{\zeta} - \tilde{\eta}}$$

where $\Gamma(z)$ is the Gamma function.

This formula is derived by taking an appropriate limit (see Section 3.17) of the following *exact* formula for the transition probability of a single particle in ASEP with step Bernoulli initial data. This means there are no particles to the left of the origin, and to the right of the origin each site has a particle with probability $\rho_+ \in (0, 1]$.

Theorem 2.6 (Tracy-Widom step Bernoulli formula [TW09b])**.** *Let $q > p$ with $q + p = 1$, $\gamma = q - p$ and $\tau = p/q$. Set*

$$\alpha = (1 - \rho_+)/\rho_+.$$

We can initially label our particles $1, 2, 3, \ldots$ by setting the leftmost to be particle 1 and the second left most to be particle 2, and so on. Let $\mathbf{x}(t, m)$ denote the location of particle m at time t. Then for $m > 0$, $t \geq 0$ and $x \in \mathbb{Z}$,

$$P(\mathbf{x}(\gamma^{-1}t, m) \leq x) = \int_{S_{\tau^+}} \frac{d\mu}{\mu} \prod_{k=0}^{\infty} (1 - \mu\tau^k) \det(I + \mu J_\mu^\Gamma)_{L^2(\Gamma_\eta)} \tag{102}$$

where S_{τ^+} is a circle centered at zero of radius strictly between τ and 1, and where the kernel of the determinant is given by

$$J_\mu^\Gamma(\eta, \eta') = \int_{\Gamma_\zeta} \exp\{\Psi(\zeta) - \Psi(\eta')\} \frac{f(\mu, \zeta/\eta')}{\eta'(\zeta - \eta)} \frac{g(\eta')}{g(\zeta)} d\zeta \tag{103}$$

where

$$\begin{aligned} f(\mu, z) &= \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \tau^k \mu} z^k \\ \Psi(\zeta) &= \Lambda(\zeta) - \Lambda(\xi) && (104) \\ \Lambda(\zeta) &= -x \log(1 - \zeta) + \frac{t\zeta}{1 - \zeta} + m \log \zeta \\ g(\zeta) &= \prod_{n=0}^{\infty} (1 + \tau^n \alpha \zeta). && (105) \end{aligned}$$

The contours are as follows: η and η' are on Γ_η, a circle symmetric about the real axis and intersecting it at $-\alpha^{-1}+2\delta$ and $1-\delta$ for δ small. And the ζ integral is on Γ_ζ, a circle of diameter $[-\alpha^{-1}+\varsigma, 1+\varsigma]$. One should choose ς so as to ensure that $|\zeta/\eta| \in (1,\tau^{-1})$. This choice of contour avoids the poles of the new infinite product which are at $-\alpha^{-1}\tau^{-n}$ for $n \geq 0$. Note that one can take δ to depend on ϵ.

2.7. **KPZ universality, or universality of KPZ?.** All of the models described in this section, as well as the KPZ equation, are believed to be members of the *KPZ universality class*, in the sense that they should have the scaling exponent $z = 3/2$, and, at a more refined level, the correct fluctuations (GUE/Airy$_2$, GOE/Airy$_1$, Baik-Rains/Brownian motion) at the scale (31) depending on the initial conditions (curved, flat, equilibrium). We have sketched above what is proved for special cases.

There is another, much more restrictive, type of universality, that of the KPZ equation itself. As we will see, it arises when one takes a continuum limit of cutoff models, while making at the same time a critical adjustment of the non-linearity. Not all models have such an adjustable non-linearity. Asymmetric exclusion does, the parameter being $q-p$. There is also a large class of generalized asymmetric exclusion models, which do [GJ10]. All directed polymers do, the parameter being β. LPP does not, and, as far as we can tell, PNG does not. These limits are described in Section 3.12. The exact results for KPZ then mean that we can obtain weak versions of some of the universality conjectures. This is the advantage of the KPZ equation over other models in the universality class.

2.8. **KPZ fixed point.** All the fluctuations, for all these models, are observed under the rescaling,

$$h_\epsilon(t,x) = (R_\epsilon h)(t,x) := \epsilon^{1/2} h(\epsilon^{-3/2}t, \epsilon^{-1}x) \tag{106}$$

after subtraction of appropriately diverging quantities.

If we started with the standard KPZ, this would satisfy the KPZ with renormalized coefficients,

$$\partial_t h_\epsilon = \tfrac{1}{2}(\partial_x h_\epsilon)^2 + \epsilon^{1/2}\tfrac{1}{2}\partial_x^2 h_\epsilon + \epsilon^{1/4}\xi. \tag{107}$$

Now we ask what happens as $\epsilon \to 0$. The limiting process should contain all the fluctuation behaviour we have observed so far. It is the fixed point of the renormalization (106). Presumably, it contains a lot of the integrable structure. So far, we do not know much about it. The rest of this section consists of conjectures [CQ11b].

We define the *KPZ fixed point* $\mathfrak{h}$ as the $\epsilon \to 0$ limit of the properly centered process $\bar{h}_\epsilon$. Note that we do not know in general that such a limit exists[4]. We now give a conjectural construction of the limiting object.

Let $h(u,y;t,x)$ be the solution of (1) for times $t > u$ started at time u with a delta function at y, all using the same noise. To center, set $\bar{h}(u,y;t,x) = h(u,y;t,x) - \frac{t-u}{24} - \log\sqrt{2\pi(t-u)}$ and define A_1 by

$$\bar{h}(u,y;t,x) = -\tfrac{(x-y)^2}{2(t-u)} + A_1(u,y;t,x).$$

After the rescaling (106),

$$R_\epsilon \bar{h}(u,y;t,x) = -\tfrac{(x-y)^2}{2(t-u)} + A_\epsilon(u,y;t,x)$$

[4] Partial results have been obtained for Poissonian last passage percolation [CP11]. There one can obtain the necessary tightness, which means that such an object exists. What is missing is its uniqueness, and, more generally the universality.

where $A_\epsilon = R_\epsilon A_1$. The R_ϵ acts on the two pairs of variables here. As $\epsilon \to 0$, $A_\epsilon(u, y; t, x)$ converges to the *Airy sheet* $\mathcal{A}(u, y; t, x)$. In each spatial variable it is an Airy_2 process. It has several nice properties:

(1) *Independent increments.* $\mathcal{A}(u, y; t, x)$ is independent of $\mathcal{A}(u', y; t', x)$ if $(u, t) \cap (u', t') = \emptyset$;
(2) *Space and time stationarity.* $\mathcal{A}(u, y; t, x) \overset{\text{dist}}{=} \mathcal{A}(u+h, y; t+h, x) \overset{\text{dist}}{=} \mathcal{A}(u, y+z; t, x+z)$;
(3) *Scaling.* $\mathcal{A}(0, y; t, x) \overset{\text{dist}}{=} t^{1/3}\mathcal{A}(0, t^{-2/3}y; 1, t^{-2/3}x)$;
(4) *Semi-group property.* For $u < s < t$,

$$\mathcal{A}(u, y; t, x) = \sup_{z\in\mathbb{R}} \big\{ \tfrac{(x-y)^2}{2(t-u)} - \tfrac{(z-y)^2}{2(s-u)} - \tfrac{(x-z)^2}{2(t-s)} + \mathcal{A}(u, y; s, z) + \mathcal{A}(s, z; t, x) \big\}. \tag{108}$$

Using $\mathcal{A}(u, y; t, x)$ we construct the KPZ fixed point $\mathfrak{h}(t, x)$. By the Hopf-Cole transformation and the linearity of the stochastic heat equation, the rescaled solution of (1) with initial data h^0 is

$$R_\epsilon h(t, x) = \epsilon^{1/2} \ln \int e^{\epsilon^{-1/2}\{-\frac{(x-y)^2}{2t} + A_\epsilon(0,y;t,x) + R_\epsilon h^0(y)\}} dy. \tag{109}$$

If we choose initial data h^0_ϵ so that $R_\epsilon h^0_\epsilon$ converges to a fixed function f in the limit, we can use Laplace's method to evaluate $\mathfrak{h}(t, x) = \lim_{\epsilon\to 0} R_\epsilon h(t, x) = T_{0,t}f(x)$ where

$$T_{u,t}f(x) := \sup_y \Big\{ -\tfrac{(x-y)^2}{2(t-u)} + \mathcal{A}(u, y; t, x) + f(y) \Big\}. \tag{110}$$

The operators $T_{u,t}$, $0 < u < t$ form a semi-group, i.e. $T_{u,t} = T_{u,s}T_{s,t}$ which is stationary with independent increments and $T_{0,t} \overset{\text{dist}}{=} R^{-1}_{t^{-2/3}} T_{0,1} R_{t^{-2/3}}$.

By the Markov property, the joint distribution of the marginal spatial process of $\mathfrak{h}$ (for initial data f) at a set of times $t_1 < t_2 < \cdots < t_n$ is given by

$$(\mathfrak{h}(t_1), \ldots, \mathfrak{h}(t_n)) \overset{\text{dist}}{=} (T_{0,t_1}f, \ldots, T_{t_{n-1},t_n} \cdots T_{0,t_1}f).$$

The process of randomly evolving functions can be thought of as a high dimensional analogue of Brownian motion (with state space Brownian motions!), and the $T_{t_i,t_{i+1}}$ as analogous to the independent increments.

The solution of (1) corresponds to the free energy of a directed random polymer $x(s)$, $u < s < t$ starting at y and ending at x, with quenched random energy

$$H(x(\cdot)) = \int_u^t \{|\dot{x}(s)|^2 - \xi(s, x(s))\} ds. \tag{111}$$

Under the rescaling (7) this probability measure on paths converges to the *polymer fixed point*; a continuous path $\pi_{u,y;t,x}(s)$, $u \le s \le t$ from y to x which at discrete times $u = s_0 < \cdots s_{m-1} < t$ is given by the argmax over $x_0, \ldots, x_{m-1}$ of

$$(T_{u,s_1}\delta_y)(x_1) + (T_{s_1,s_2}\delta_{x_1})(x_2) + \cdots + (T_{s_{m-1},t}\delta_{x_{m-1}})(x). \tag{112}$$

This is the analogue in the present context of the minimization of the action and the polymer fixed point paths are analogous to characteristics in the randomly forced Burger's equation. One might hope to take the analogy farther and find a limit of the renormalizations of (111), and minimize it to find that path $\pi_{u,y;t,x}$. However, the limit does not appear to exist, so one has to be satisfied with the limiting paths themselves. The path $\pi_{0,y;t,x}$ turns out to be Hölder continuous with exponent $1/3-$, as compared to Brownian motion where the Hölder exponent is $1/2-$. As the mesh of times is made finer, a limit

$\mathcal{E}(\pi_{0,y;t,x})$ of (112) does exist, and through it we can write the time evolution of the KPZ fixed point in terms of the polymer fixed point through the analogue of the Lax-Oleinik variational formula,

$$\mathfrak{h}(t,x) = \sup_{y\in R}\{\mathcal{E}(\pi_{0,y;t,x}) + f(\pi_{0,y;x,t}(0))\}. \tag{113}$$

The KPZ fixed point, Airy sheet, and polymer fixed point are universal and will arise in random polymers, last passage percolation and growth models – anything in the KPZ universality class. Just as for (1), at the microscopic scale, approximate versions of the variational problem (110) hold, becoming exact as $\epsilon \to 0$.

3. A mathematical introduction

3.1. **White noise and stochastic integration in $1+1$ dimensions.**

In the following sections we give a rigorous introduction to white noise and the stochastic heat equation. Following that we sketch the proofs that discrete models converge to the stochastic heat equation, and the corresponding limit of the Tracy-Widom step Bernoulli formula, which then leads to the edge crossover distributions. So we now switch from heuristics to rigorous mathematics.

White noise $\xi(t,x)$, $t \geq 0$, $x \in \mathbb{R}$ is the distribution valued Gaussian process with mean zero and covariance

$$E[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y). \tag{114}$$

More precisely, for any smooth function f of compact support we can write[5]

$$\int_{\mathbb{R}_+\times\mathbb{R}} f(t,x)\xi(t,x)dxdt, \tag{115}$$

and the random variables $\{\int_{\mathbb{R}_+\times\mathbb{R}} f(t,x)\xi(t,x)dxdt\}_{f\in L^2(\mathbb{R}_+\times\mathbb{R})}$ are jointly Gaussian with mean zero and covariance

$$E[\int_{\mathbb{R}_+\times\mathbb{R}} f_1(t,x)\xi(t,x)dxdt \int_{\mathbb{R}_+\times\mathbb{R}} f_2(t,x)\xi(t,x)dxdt] = \int_{\mathbb{R}_+\times\mathbb{R}} f_1(t,x)f_2(t,x)dxdt. \tag{116}$$

There are many ways to construct it. One way is to choose an orthonormal basis $f_1, f_2, \ldots$ of $L^2(\mathbb{R}_+\times\mathbb{R})$, and independent Gaussian random variables $Z_1, Z_2, \ldots$, each with mean zero and variance one, and write

$$\xi(t,x) = \sum_{n=1}^{\infty} Z_n f_n(t,x). \tag{117}$$

One can check the resulting object makes sense as an element of the negative Sobolev space $H_{-1-\delta,\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R})$ for $\delta > 0$.

We assume that our white noise has been constructed on a probability space $(\Omega, \mathscr{F}, P)$. This could be in the way described above, or some other way. We now start to construct stochastic integrals. For non-random functions this is easy. We have done it in (115) for smooth functions of compact support in $\mathbb{R}_+\times\mathbb{R}$. If $f \in L^2(\mathbb{R}_+\times\mathbb{R})$ then we define

$$\int_{\mathbb{R}_+\times\mathbb{R}} f(t,x)\xi(t,x)dxdt \tag{118}$$

[5] We will always use the natural notation $\int fg$ when f is a smooth test function and g is a distribution.

by approximation. Indeed, there are smooth functions f_n of compact support in $\mathbb{R}_+ \times \mathbb{R}$, with

$$\int_{\mathbb{R}_+\times\mathbb{R}} |f_n(t,x) - f(t,x)|^2 dxdt \to 0. \tag{119}$$

The f_n are therefore Cauchy in $L^2(\mathbb{R}_+ \times \mathbb{R})$. But by (116) we have

$$E[(\int_{\mathbb{R}_+\times\mathbb{R}} (f_n(t,x) - f_m(t,x))\xi(t,x)dxdt)^2] = \int_{\mathbb{R}_+\times\mathbb{R}} |f_n(t,x) - f_m(t,x)|^2 dxdt \tag{120}$$

so $\int_{\mathbb{R}_+\times\mathbb{R}} f_n(t,x)\xi(t,x)dxdt$ is a Cauchy sequence in $L^2(\Omega, \mathscr{F}, P)$, and therefore has a limit. Hence (118) exists as a limiting object in $L^2(\Omega, \mathscr{F}, P)$.

However, we will also need to integrate random functions, in particular functions which depend on the white noise ξ. It is here that things become a little more complicated, and, as in the one dimensional case, one has to make a choice of integral.

We now define the stochastic integral we need. Because we are dealing with a parabolic type stochastic partial differential equation, the time and space directions are treated differently. Our stochastic integral is basically the standard Itô integral in the time variable. As in the one dimensional case we start with the simplest integrators and build the integral by approximation.

For smooth functions φ on $\mathbb{R}$ with compact support and $t > 0$ we can define

$$\int_{\mathbb{R}_+\times\mathbb{R}} \mathbf{1}_{(0,t]}(s)\varphi(x)\xi(s,x)dxds. \tag{121}$$

For each fixed φ, it is a Brownian motion in t, with variance $\int \varphi^2(x)dx$, i.e.the Gaussian process with mean zero and covariance from (116),

$$E[\int \mathbf{1}_{(0,t_1]}(s)\varphi(x)\xi(s,x)dxds \int \mathbf{1}_{(0,t_2]}(s)\varphi(x)\xi(s,x)dxds] = \min(t_1,t_2) \int \varphi^2(x)dx. \tag{122}$$

Let $\mathscr{F}_0 = \emptyset$ and for each $t > 0$, define $\mathscr{F}_t$ to be the $\sigma-$field generated by

$$\left\{ \int \mathbf{1}_{(0,s]}(u)\varphi(x)\xi(u,x)dxdu \right\}, 0 \le s \le t, \ \varphi \text{ a smooth function of compact support on } \mathbb{R}.$$

$\mathscr{F}_s \subseteq \mathscr{F}_t$ if $s \le t$, i.e. $\mathscr{F}_t$ is a filtration in $\mathscr{F}$.

Let $\mathcal{S}$ be the set of functions of the form

$$f(t,x,\omega) = \sum_{i=1}^n X_i(\omega)\mathbf{1}_{(a_i,b_i]}(t)\varphi_i(x) \tag{123}$$

where X_i is a bounded random variable measurable with respect to $\mathscr{F}_{a_i}$, $0 \le a_i \le b_i < \infty$ and φ_i are smooth function of compact support on $\mathbb{R}$.

For $f \in \mathcal{S}$ we define the stochastic integral as

$$\int_{\mathbb{R}_+\times\mathbb{R}} f(t,x)\xi(t,x)dxdt = \sum_{i=1}^n X_i \int_{\mathbb{R}_+\times\mathbb{R}} \mathbf{1}_{(a_i,b_i]}(t)\varphi_i(x)dxdt. \tag{124}$$

the important point is that the time increment sticks out into the future. For such integrators one easily checks the linearity of the integral and the isometry

$$E\left[\left(\int_{\mathbb{R}_+\times\mathbb{R}} f(t,x)\xi(t,x)dxdt\right)^2\right] = \int_{\mathbb{R}_+\times\mathbb{R}} E[f^2(t,x)]dxdt. \tag{125}$$

Let $\mathscr{P}$ denote the sub-$\sigma-$field of $\mathscr{B}(\mathbb{R}_+\times\mathbb{R})\times\mathscr{F}$ generated by $\mathcal{S}$. Let $L^2(\mathbb{R}_+\times\mathbb{R}\times\Omega,\mathscr{P})$ be the linear subspace of $f(t,x,\omega)$ measurable with respect to $\mathscr{P}$ and square integrable. These will be our integrators. Like all Itô integrals the key point is that they are in some sense "non-anticipating", i.e. $f(t,x,\omega)$ only depends on the information $\mathscr{F}_t$ up to time t. We construct the stochastic integral for them through the isometry, by approximation, so we need the following

Lemma 3.1. *$\mathcal{S}$ is dense in $L^2(\mathbb{R}_+\times\mathbb{R}\times\Omega,\mathscr{P})$.*

Proof. The proof is the same as the one dimensional case. If $f\in L^2(\mathbb{R}_+\times\mathbb{R}\times\Omega,\mathscr{P})$, then $f_n=f\mathbf{1}_{|f|\le n,t\le n}$ are bounded, supported on $t\le n$ and measurable with respect to $\mathscr{P}$ and $\|f-f_n\|_{L^2(\mathbb{R}_+\times\mathbb{R}\times\Omega,\mathscr{P})}\to 0$ by the monotone convergence theorem. If f is bounded and measurable with respect to $\mathscr{P}$, and supported on $t\le t_0<\infty$ then

$$f_n(t,x,\omega)=\sum_{k=0}^{\infty}\mathbf{1}_{[\frac{k}{2^n},\frac{k+1}{2^n})}(s)\tfrac{1}{2^n}\int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}}f(s,x,\omega)ds \tag{126}$$

are in $\mathcal{S}$ and $\|f-f_n\|_{L^2(\mathbb{R}_+\times\mathbb{R}\times\Omega,\mathscr{P})}\to 0$ by the Lebesgue differentiation and bounded convergence theorems. $\square$

Consequently, if $f\in L^2(\mathbb{R}_+\times\mathbb{R}\times\Omega,\mathscr{P})$ we can choose $f_n\in\mathcal{S}$ with $f_n\to f$ in $L^2(\mathbb{R}_+\times\mathbb{R}\times\Omega,\mathscr{P})$. By the isometry (125),

$$I_n(\omega)=\int_{\mathbb{R}_+\times\mathbb{R}}f_n(t,x,\omega)\xi(t,x,\omega)dxdt$$

is a Cauchy sequence in $L^2(P)$, i.e. $\lim_{n,m\to\infty}E[(I_n-I_m)^2]=0$. Hence there is a limit point $I\in L^2(P)$ which we call the stochastic integral $\int_{\mathbb{R}_+\times\mathbb{R}}f(t,x,\omega)\xi(t,x,\omega)dxdt$. It is linear in f and we have the Itô isometry (125).

We will need later

Lemma 3.2 (Burkholder's inequality)**.** *For $p\ge 2$ there is a $C_p<\infty$ such that*

$$E[|\int_{\mathbb{R}_+\times\mathbb{R}}f(t,x,\omega)\xi(t,x,\omega)dxdt|^p]\le CE[|\int_{\mathbb{R}_+\times\mathbb{R}}f^2(t,x,\omega)dxdt|^{p/2}] \tag{127}$$

3.2. **Wiener chaos.** We define multiple stochastic integrals,

$$\begin{aligned}&\int_{\mathbb{R}_+^k}\int_{\mathbb{R}^k}f(t_1,\dots,t_k,x_1\dots,x_k)\xi(t_1,x_1)\cdots\xi(t_k,x_k)dx_1\cdots dx_kdt_1\cdots dt_k\\&=\int_{\mathbb{R}_+^k}\int_{\mathbb{R}^k}f(\mathbf{t},\mathbf{x})\xi^{\otimes\mathbf{k}}(\mathbf{t},\mathbf{x})\mathbf{dt}\,\mathbf{dx}.\end{aligned}$$

The construction is similar to the $k=1$ case except that care must be taken with respect to integration along the "diagonals". f is symmetric if $f(\mathbf{t},\mathbf{x})=\mathbf{f}(\pi\mathbf{t},\pi\mathbf{x})$ for all $(\mathbf{t},\mathbf{x})\in\mathbb{R}_+^{\mathbf{k}}\times\mathbb{R}^{\mathbf{k}}$ and $\pi\in S_k$, the group of permutations on $\{1,2,\dots,k\}$. We let $\hat{L}^2(\mathbb{R}_+^k\times\mathbb{R}^k)$ denote the subspace of symmetric functions in L^2. As in the $k=1$ case it is enough to define the integral on a dense subset from which it can be linearly extended (in a unique way) to the entire space. One dense subset is linear combinations of functions of the form

$$f(\mathbf{t},\mathbf{x})=\sum_{\pi\in\mathbf{S_k}}\prod_{\mathbf{j=1}}^{\mathbf{k}}\mathbf{1}_{(\mathbf{t}_{\pi\mathbf{j}},\mathbf{x}_{\pi\mathbf{j}})\in\mathbf{A_j}}, \tag{128}$$

where the $A_j, j = 1, \dots, k$ are bounded disjoint rectangles in $\mathbb{R}_+ \times \mathbb{R}$. We define

$$\int_{\mathbb{R}_+^k} \int_{\mathbb{R}^k} f(\mathbf{t},\mathbf{x})\xi^{\otimes \mathbf{k}}(\mathbf{t},\mathbf{x})\mathbf{dt}\,\mathbf{dx} = \mathbf{k}! \prod_{\mathbf{j}=\mathbf{1}}^{\mathbf{k}} \int_{\mathbf{A_j}} \xi(\mathbf{t},\mathbf{x})\mathbf{dt}\mathbf{dx}. \tag{129}$$

It is standard to show that there exists a unique linear extension from linear combinations of functions of the form (128) onto $\hat{L}^2$ such that the stochastic integral is a mean zero Gaussian with variance $||f||^2_{L^2}$, and the covariance structure is

$$E\int_{\mathbb{R}_+^k} \int_{\mathbb{R}^k} f(\mathbf{t},\mathbf{x})\xi^{\otimes \mathbf{k}}(\mathbf{t},\mathbf{x})\mathbf{dt}\,\mathbf{dx} \int_{\mathbb{R}_+^{\mathbf{j}}} \int_{\mathbb{R}^{\mathbf{j}}} \mathbf{g}(\mathbf{t},\mathbf{x})\xi^{\otimes \mathbf{j}}(\mathbf{t},\mathbf{x})\mathbf{dt}\,\mathbf{dx} = \langle f, g\rangle_{L^2([0,1]^k\times\mathbb{R}^k)} \mathbf{1}_{j=k}. \tag{130}$$

There is an isometry between $L^2(\Omega, \mathscr{F}, P)$ and $\bigoplus_{k=0}^\infty \hat{L}^2(\mathbb{R}_+^k \times \mathbb{R}^k)$, i.e. given a random variable $X \in L^2(\Omega, \mathscr{F}, P)$, there are $f_k \in \hat{L}^2(\mathbb{R}_+^k \times \mathbb{R}^k), k = 0, 1, 2, \dots$, with

$$X = \sum_{k=0}^\infty \int_{\mathbb{R}_+^k} \int_{\mathbb{R}^k} f_k(\mathbf{t},\mathbf{x})\xi^{\otimes \mathbf{k}}(\mathbf{t},\mathbf{x})\mathbf{dt}\,\mathbf{dx}.$$

Here f_0 is simply the constant EX, with the convention $\int f_0 = f_0$. By (130),

$$E[X^2] = \sum_{k=0}^\infty ||f_k||^2_{L^2(\mathbb{R}_+^k\times\mathbb{R}^k)}.$$

This Wiener chaos representation can be very useful. But the problem is that if φ is non-linear, then knowing the chaos representation of X tells us nothing about the chaos representation of $\varphi(X)$. For example, in Section 3.3 we will see that the chaos series for the solution $z(t, x)$ of the stochastic heat equation is particularly simple. But unfortunately, we do not know the chaos series for the main quantity of interest in these notes, $\log z(t, x)$, the Hopf-Cole solution of KPZ.

3.3. **The stochastic heat equation.** The stochastic heat equation is

$$\begin{aligned} &\partial_t z = \tfrac{1}{2}\partial_x^2 z + \xi z, \qquad t > 0,\ x \in \mathbb{R}, \\ &z(0, x) = z_0(x), \end{aligned} \tag{131}$$

where $\xi(t, x)$ is space-time white noise. z_0 could be taken to be random, but we will always assume that it and the white noise ξ are independent.

All our physical problems will only involve z_0 which are positive in some sense. However, for purposes of proving uniqueness we will have to take differences. The solvability will depend on the initial data, and will have to include objects like exponential Brownian motions, as well as delta function initial data. A reasonable class which includes everything we want is $z_0(x)$ such that $\int_A z_0(x)dx$ is a signed measure in A satisfying for some $c < \infty$,

$$E[\sup_{A\subset[-n,n]} \int_A z_0(x)dx] < ce^{cn}. \tag{132}$$

This class is preserved by the heat semigroup.

3.4. **Mild solutions.** To make sense of the equation, we rewrite it in Duhamel form

$$z(t,x) = \int_{\mathbb{R}} p(t,x-y)z_0(y)dy + \int_0^t \int_{\mathbb{R}} p(t-s,x-y)z(s,y)\xi(s,y)dyds \tag{133}$$

using the kernel of the heat equation

$$p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}. \tag{134}$$

If $z(t,x)$ is progressively measurable with

$$\int_0^t \int_{\mathbb{R}} p^2(t-s,x-y)E[z^2(s,y)]dyds < \infty \tag{135}$$

then the stochastic integral in (133) makes sense as an element of $L^2(P)$ as shown in the previous section. Such a $z(t,x)$ for which equality holds in (133) for all $0 \le t \le T$ and $x \in \mathbb{R}$ will be called a *mild solution* of the stochastic heat equation (131).

Of course we need to know such a solution exists. There are many ways to do it, but an easy way is Picard iteration: Let $z^0(t,x) = 0$ and

$$z^{n+1}(t,x) = \int_{\mathbb{R}} p(t,x-y)z_0(y)dy + \int_0^t \int_{\mathbb{R}} p(t-s,x-y)z^n(s,y)\xi(s,y)dyds. \tag{136}$$

They are progressively measurable by construction. Let $\bar{z}^n(t,x) = z^{n+1}(t,x) - z^n(t,x)$. Then

$$\bar{z}^{n+1}(t,x) = \int_0^t \int_{\mathbb{R}} p(t-s,x-y)\bar{z}^n(s,y)\xi(s,y)dyds. \tag{137}$$

So

$$E[|\bar{z}^{n+1}(t,x)|^2] = \int_0^t \int_{\mathbb{R}} p(t-s,x-y)E[|\bar{z}^n(t,x)|^2]dyds. \tag{138}$$

Letting $f^n(t) = \sup_{x,s\in[0,t]} E[|\bar{z}^n(s,x)|^2]$ we have, after integrating the heat kernel,

$$f^{n+1}(t) \le C \int_0^t \frac{f^n(s)}{\sqrt{t-s}} ds. \tag{139}$$

Iterating the inequality,

$$f^{n+1}(t) \le C \int_0^t \int_0^s \frac{f^{n-1}(u)}{\sqrt{(t-s)(s-u)}} duds. \tag{140}$$

Changing the order of integration

$$f^{n+1}(t) \le C \int_0^t \int_u^t \frac{f^{n-1}(u)}{\sqrt{(t-s)(s-u)}} dsdu = C' \int_0^t f^{n-1}(u)du. \tag{141}$$

Now Gronwall's inequality implies that $f^n(t) \le (C't)^{n/2}/(n/2)!$, so there is a limit $z(t,x)$ which is progressively measurable and solves the equation. Note that the same argument also proves that $z(t,x)$ is unique.

3.5. A priori estimate.

Lemma 3.3. *Let $z(t,x)$ satisfy (133) with z_0 as indicated. Then*

$$E[z^2(t,x)] \leq ce^{ct}. \tag{142}$$

Let $z(t,x)$ satisfy (133) with $z_0 = \delta_0$ (or, in general, localized initial data). Then there exists a $c = c(T)$ such that for all $0 < t \leq T$, for all

$$E[z^2(t,x)] \leq cp^2(t,x). \tag{143}$$

Proof. Without loss of generality we can assume the initial data is non-random because we could always take the conditional expectation given $\mathscr{F}_0$, and then take a further expectation to get the result. Then, squaring (133) and taking expectation we have

$$E\left[z^2(t,x)\right] = \left(\int_{\mathbb{R}} p(t,x-y)z_0(y)dy\right)^2 + \int_0^t \int_{\mathbb{R}} p^2(t-s,x-y)E\left[z^2(s,y)\right]dyds. \tag{144}$$

Iterating the equation, we obtain

$$E\left[z^2(t,x)\right] = \sum_{n=0}^{\infty} E[I_n(t,x)] \tag{145}$$

where, for $\Delta_n(t) = \{0 \leq t_n \leq t_{n-1} \leq \cdots \leq t_0 = t\}$, $x_0 = x$

$$I_n(t,x) = \int_{\Delta_n(t)} \int_{\mathbb{R}^n} \prod_{i=0}^{n-1} p^2(t_i - t_{i+1}, x_i - x_{i+1}) \left(\int_{\mathbb{R}} p(t_n, x_n - y)z_0(y)dy\right)^2 \prod_{i=1}^{n} dx_i dt_i. \tag{146}$$

Now we use the fact that for $s < u < t$

$$\int_{\mathbb{R}} p^2(t-u,x-z)p^2(u-s,z-y)dz = \sqrt{\tfrac{t-s}{4\pi(t-u)(u-s)}}\,p^2(t-s,x-y) \tag{147}$$

to simplify

$$\begin{aligned}
I_n(t,x) &= \frac{1}{\Gamma(\frac{n-1}{2})} \int_{\Delta_n(t)} \prod_{i=1}^{n} \frac{dt_i}{\sqrt{4\pi(t_{i-1}-t_i)}} \int_{\mathbb{R}} p(\tfrac{t_0-t_n}{2}, x_0 - x_n) \left(\int_{\mathbb{R}} p(t_n, x_n - y)z_0(y)dy\right)^2 dx_n \\
&= \frac{2^{-n+\frac{1}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^t \int (t-s)^{\frac{n}{2}}\, p^2(t-s,x-z) \left(\int_{\mathbb{R}} p(s,z-y)z_0(y)dy\right)^2 dzds \\
&\leq C(\tfrac{t}{4})^{n/2}(n!)^{-1/2} \int_0^t \int p^2(t-s,x-z) \left(\int_{\mathbb{R}} p(s,z-y)z_0(y)dy\right)^2 dzds.
\end{aligned}$$

(132) implies that is a C such that

$$\limsup_{s\to 0} s^{1/2}\mathbb{E}\left[\left(\int_{\mathbb{R}} dyp(s,x-y)z_0(y)\right)^2\right] < \infty. \tag{148}$$

We obtain $E[I_n] \leq C''(\frac{t}{4})^{n/2}(n!)^{-1/2}$. Summing in n gives (142). Keeping track of the heat kernels in the last two bounds one obtain (143). ∎

3.6. **Martingale problem.** The solution $z(t,x)$ of the stochastic heat equation (131) is a random element of $C(\mathbb{R}_+, C(\mathbb{R}))$ where C means the space of continuous functions with the topology of uniform convergence on compact sets. We can also think of it through its distribution, which is a probability measure Q on $C(\mathbb{R}_+, C(\mathbb{R}))$. $C(\mathbb{R}_+, C(\mathbb{R}))$ comes equipped with a natural filtration $\mathscr{F}_t$ which is the $\sigma-$field generated by $z(s,x)$, $x \in \mathbb{R}$, $s \le t$. The martingale formulation is an alternative way to identify the probability measure Q by specifying that under it, a rich enough class of processes should be martingales with respect to $\mathcal{F}_t$.

For technical reasons we will insist that our probability measure Q satisfies for all $T > 0$

$$\sup_{s\in[0,T]} \sup_{x\in\mathbb{R}} e^{-a|x|} E^Q\left[(z(s,x))^2\right] < \infty \tag{149}$$

for some $a > 0$.

Definition 3.4. Such a probability measure Q on $(C(\mathbb{R}^+, C(\mathbb{R})), \mathscr{F})$ is a *solution of the martingale problem for the stochastic heat equation (131) with initial data* z_0 if for all smooth test functions φ on $\mathbb{R}$ with compact support, both

$$N_t(\varphi) = \int_{\mathbb{R}} \varphi(x) z(t,x)dx - \int_{\mathbb{R}} \varphi(x) z_0(x) - \tfrac{1}{2}\int_0^t \int_{\mathbb{R}} \varphi''(x) z(s,x)dxds, \tag{150}$$

and

$$\Lambda_t(\varphi) = N_t(\varphi)^2 - \int_0^t \int_{\mathbb{R}} \varphi^2(x) z^2(s,x)dxds \tag{151}$$

are local martingales under Q.

The martingale problem is extremely useful for studying asymptotic problems for stochastic processes. For example, if we want to prove a discrete KPZ model converges to Hopf-Cole solution of KPZ, we know that we really mean the convergence of the exponentiated version of the former to the solution of the stochastic heat equation. We can do it by writing down approximate versions of (150) and (151) which are martingales in the discrete model. If we can prove tightness, then the limiting process has to solve the martingale problem. The key is therefore the following uniqueness result [BG97, KS88].

Proposition 3.5. The martingale problem for the stochastic heat equation has a unique solution which coincides with the distribution of the unique strong solution $z(t,x)$ of (131).

Remark 3.6. Some readers may be more familiar with the classic martingale problem for finite dimensional stochastic differential equations, which asks that for a nice class of functions f on $\mathbb{R}^d$ (say smooth functions of compact support),

$$M_f(t) := f(x(t)) - \int_0^t Lf(x(s))ds \tag{152}$$

be martingales, where $Lf = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\partial^2_{x_i x_j} f + \sum_{i=1}^d b_i(x)\partial_{x_i} f$. Under reasonable conditions, this produces a weak solution of the stochastic differential equation $dx = b(x)dt + \sigma(x)dB$ on $\mathbb{R}^d$.

Another, older, formulation is simply to ask that this holds for the functions $f(x) = x_i$, $i = 1, \dots, d$ and $f(x) = x_i x_j$, $i,j = 1,\dots,d$ (as local martingales).

In fact, we don't even need the quadratic ones if we ask that

$$N_t(i) := x_i(t) - \int_0^t b_i(x(s))ds$$

and

$$\Lambda_t(i,j) := N_t(i)N_t(j) - \int_0^t a_{ij}(x(s))ds$$

are (local) martingales.

Of course, there is no reason to fix a special basis of $\mathbb{R}^d$, and the sum of local martingales is a local martingale. So we may as well take vectors $\varphi \in \mathbb{R}^d$ and ask that $N_t(\varphi) := \varphi \cdot x(t) - \int_0^t \varphi \cdot b(x(s))ds$ and $\Lambda_t(\varphi) = N_t^2(\varphi) - \int_0^t \varphi \cdot a(x(s))\varphi ds$ are martingales.

The two equations (150) and (151) are just the infinite dimensional version of this last criterion.

3.7. **Chaos representation.** Iterating (133) once we obtain

$$\begin{aligned} z(t,x) = &\int_{\mathbb{R}} p(t, x-y)z_0(y)dy + \int_0^t \int_{\mathbb{R}^2} p(t-s, x-y_2)p(s, y_2-y_1)z_0(y_1)\xi(s,y_1)dy_1dy_2ds \\ &+ \int_{0\le s_1<s_2\le t}\int_{\mathbb{R}^2} p(t-s_2, x-y_2)p(s_2-s_1, y_2-y_1)z(s_1,y_1)\xi(s_1,y_1)\xi(s_2,y_2)dy_1dy_2ds_1ds_2. \end{aligned}$$

Iterating this $n-1$ times we obtain,

$$\begin{aligned} z(t,x) = &\sum_{k=0}^{n-1}\int_{0\le s_1<\cdots<s_k\le t}\int_{\mathbb{R}^{k+1}} p(t-s_k, x-y_k)p(s_k-s_{k-1}, y_k-y_{k-1})\cdots p(s_2-s_1, y_2-y_1)\times \\ &\qquad \times\, p(s_1, y_1-y_0)z_0(y_0)\xi(s_1,y_1)\cdots\xi(s_k,y_k)dy_0dy_1\cdots dy_kds_1\cdots ds_k \\ &+ \int_{0\le s_1<\cdots<s_n\le t}\int_{\mathbb{R}^n} p(t-s_n, x-y_n)p(s_n-s_{n-1}, y_n-y_{n-1})\cdots p(s_2-s_1, y_2-y_1)\times \\ &\qquad \times\, z(s_1,y_1)\xi(s_1,y_1)\cdots\xi(s_n,y_n)dy_1\cdots dy_nds_1\cdots ds_n. \end{aligned}$$

Compute the L^2 norm of the nth term

$$\begin{aligned} &\int_{0\le s_1<\cdots<s_n\le t}\int_{\mathbb{R}^n} p^2(t-s_n, x-y_n)p^2(s_n-s_{n-1}, y_n-y_{n-1})\cdots p^2(s_2-s_1, y_2-y_1)\times \\ &\qquad \times\, E[z^2(s_1,y_1)]dy_1\cdots dy_nds_1\cdots ds_n. \end{aligned}$$

Use the apriori estimate Lemma 3.3 to see that the nth term goes to 0 in $L^2(P)$ as $n\to\infty$. We obtain the chaos representation

$$z(t,x) = \sum_{k=0}^{\infty}\int_{\Delta_k}\int_{\mathbb{R}^k}\int_{\mathbb{R}} \mathbf{p}_k(\mathbf{s}, y_0, \mathbf{y})z_0(y_0)dy_0\xi^{\otimes k}(\mathbf{s},\mathbf{y})d\mathbf{y}d\mathbf{s}. \tag{153}$$

where $\Delta_k = \{0\le s_1<\cdots<s_k\le t\}$ and for such an $\mathbf{s} = (s_1,\dots,s_k)$

$$\mathbf{p}_k(\mathbf{s}, y_0, \mathbf{y}) = p(t-s_k, x-y_k)p(s_k-s_{k-1}, y_k-y_{k-1})\cdots p(s_2-s_1, y_2-y_1)p(s_1, y_1-y_0) \tag{154}$$

are Brownian transition densities to be at y_i at times s_i, $i=1,\dots,k$ and end up at x at time t, given that the Brownian motion started at y_0 at time 0.

Since

$$\mathbf{p}_k^2(\mathbf{s},\mathbf{y}) = \mathbf{p}_k(\mathbf{s},\sqrt{2}\mathbf{y})\prod_{j=1}^k \frac{1}{\sqrt{2\pi(s_j-s_{j-1})}},$$

we have

$$\int_{\Delta_k}\int_{\mathbb{R}^k}\mathbf{p}_k^2(\mathbf{s},\mathbf{y})\,d\mathbf{y}\,d\mathbf{s} = (4\pi)^{-\frac{k}{2}}\int_{\Delta_k}\prod_{j=1}^{k}\frac{1}{\sqrt{s_j-s_{j-1}}}\,d\mathbf{s} = \frac{1}{2^k\Gamma\left(\frac{k}{2}+1\right)}. \tag{155}$$

The second equality comes from recognizing that the integrand is the density of a Dirichlet distribution.

So (153) is easily seen to be a valid chaos series.

The chaos series represents an alternative way to prove discrete models converge to the stochastic heat equation. For example, the directed random polymer partition functions have the form of discrete versions of the chaos series.

To illustrate the use of the chaos series, we have the following small time asymptotics for the solution of the stochastic heat equation with Dirac initial data, which is readily checked by computing the mean square of the $k \geq 2$ terms in the chaos series (153).

Proposition 3.7. Let $z(t,x)$ be the solution of (131) with $z(0,x) = \delta_0(x)$. As $t \searrow 0$,

$$z(t,x) = p(t,x) + t^{1/4}\zeta(t,x) + \mathcal{O}(t^{1/2}) \tag{156}$$

where the process $\zeta(t,x) = t^{-1/4}\int_0^t\int_{\mathbb{R}} p(t-s,x-y)p(s,y)\xi(s,y)dyds$ is Gaussian mean zero, with finite limiting covariance as $t \searrow 0$.

So for small time, the heat part of the equation dominates.

Note that on the physical side this is related to the picture of the weakly asymmetric scaling being critical between the KPZ and the Edwards-Wilkinsen (EW) universality classes. The relation between t and β in the continuum random polymer is $t \sim \beta^4$. So $t \to 0$ is the same as $\beta \to 0$, and as the asymmetry goes away, one finds oneself in the EW class. Another way to think about it is to put a small parameter in front of the non-linearity in KPZ and send it to zero, to see the Gaussian fluctuations.

3.8. **Regularity.**

Theorem 3.8. *Let $z(t,x)$ be the solution to (131) with $z_0(x) \geq 0$ satisfying (132). For any $\alpha < 1/2$ and $\beta < 1/4$, and any $\delta > 0$, $T < \infty$,*

$$\lim_{\lambda\to\infty} P\left(\sup_{\delta\leq s<t\leq T,\ |x|,|y|\leq B,}\frac{|z(t,x)-z(s,y|}{|t-s|^\beta+|x-y|^\alpha}\geq\lambda\right) = 0 \tag{157}$$

This can be proved with the methods of Section 3.14. Since it is almost the same, we will not repeat it here. There are many references, e.g. [Wal86], [Kho].

3.9. **Comparison.** We are interested in studying stochastic partial differential equations of the type

$$\begin{aligned} &\partial_t z = \tfrac{1}{2}\partial_x^2 z + f(z) + \sigma(z)\xi, \qquad t>0,\ x\in\mathbb{R},\\ &z(0,x) = z_0(x). \end{aligned} \tag{158}$$

Of course, this means the Duhamel version

$$z(t,x) = \int_{\mathbb{R}} p(t,x-y)z_0(y)dy + \int_0^t\int_{\mathbb{R}} p(t-s,x-y)f(z(s,y))dyds \tag{159}$$

$$+\int_0^t\int_{\mathbb{R}} p(t-s,x-y)\sigma(z(s,y))\xi(s,y)dyds. \tag{160}$$

If f and σ are Lipschitz, then the existence and uniqueness theorem still applies (see [DKM+09]).)

There are not many general methods for studying stochastic partial differential equations. One very disturbing thing to keep in mind is that if z satisfies (158) and Φ is non-linear in the sense that $\Phi''(z) \neq 0$ then $\partial_t \Phi(z)$ does *not* satisfy a stochastic partial differential equation. The reason is that unlike the case of finite dimensional stochastic differential equations, the Itô correction is always infinite. One way out is to consider *coloured noises* where the correlations $\langle \xi(t,x), \xi(s,y) \rangle = \phi(y-x)\delta(t-s)$ are more reasonable in space. This is much the same thing as discretizing the equation,

$$\partial_t z_{\frac{i}{n}} = \tfrac{1}{2}[z_{\frac{i+1}{n}} - 2z_{\frac{i}{n}} z_{\frac{i-1}{n}}] + f(z_{\frac{i}{n}}) + n^{1/2}\sigma(z_{\frac{i}{n}}) dB_{\frac{i}{n}}. \tag{161}$$

One of the few general facts that are available is the *comparison principle.*

Proposition 3.9. Let $z_1(t,x), z_2(t,x)$ be two solutions of (158) with $f_1(z) \leq f_2(z)$ and $\sigma_1(z) = \sigma_2(z)$ Lipschitz functions, both using the same white noise ξ and with initial data

$$z_1(0,x) \leq z_2(0,x), \qquad x \in \mathbb{R}$$

Then, with probability one, for all $t \geq 0$,

$$z_1(t,x) \leq z_2(t,x), \qquad x \in \mathbb{R}.$$

The proof is via approximation by discretizations (161), for which the comparison is almost obvious. Note that the Lipschitz assumption is only used in the uniqueness. In its absence one still has versions for which the comparison is true, which can be sufficient to prove distributional statements about the solutions (see [MMQ11] for an example.) In the case of the stochastic heat equation (131), we can also prove Proposition 3.9 by approximation with asymmetric exclusion (see Section 205). The comparison at the discrete level follows from the basic coupling for exclusion.

3.10. **Positivity.**

Theorem 3.10 (Mueller 91 [Mue91])**.** *Let $z(t,x)$ be the solution to (131) with $z_0(x) \geq 0$ and $\int z_0(x)dx > 0$. Then for all $t > 0$, $z(t,x) > 0$ for all $x \in \mathbb{R}$ with probability one.*

Remark 3.11. It is also true that $z(t,x) > 0$ for all $x \in \mathbb{R}$ and $t > 0$ with probability one, but we only give the proof of the weaker statement.

Remark 3.12. We could also start with a non-trivial positive measure. In a small time the heat part of the semigroup sends us in to the situation above.

Remark 3.13. If the theorem seems obvious, note that it is only *just* true. If $\frac{1}{2} < \gamma < 1$ then if z_0 has compact support, the solution of $\partial_t z = \frac{1}{2}\partial_x^2 z + \xi z^\gamma$ does as well [Isc88]. When z is small, it is a battle between the heat equation trying to spread the stuff out and the random term killing the stuff. If $\gamma < 1$ then the random term is too large and it wins. If $\gamma \geq 1$ the random term is too small and the heat part wins.

Let

$$N(t,x) = \int_0^t \int_{\mathbb{R}} p(t-s, x-y) f(s,y) \xi(s,y) dy ds. \tag{162}$$

If $f(s,y) = z(s,y)$ then this is the noise term in (133). We need to show that it doesn't win against the heat equation smoothing, so we need an estimate of how big it is. It is a random function of t and x (a random field) and we need to control the maximum. If the random f were not in the integrand then we just have a mean zero Gaussian field. We compute the covariance, and knowing this, the maximum can be controlled in terms of a typical point. The main point of the following large deviations estimate is that roughly the same thing is true for (162).

Lemma 3.14. *Suppose that $f \in \mathscr{P}$ with $|f(s,y)| \le K$, then, for $r < \infty$ there exists $0 < C < \infty$ such that*

$$P(\sup_{\substack{0\le t\le T\\ |x|\le r}} |N(t,x)| \ge \lambda) \le C \exp\left\{-C\lambda^2 T^{-1/2} K^{-2}\right\}. \tag{163}$$

A proof of the lemma can be found on page 30 of [DKM+09].

We sketch the original proof of Theorem 3.10 due to Mueller [Mue91]. It would be nice to have alternative arguments.

Proof of Theorem 3.10. We have $z_0(x) \ge \gamma \mathbf{1}_{[b-a,b+a]}(x)$ for some $a > 0$, b and $\gamma > 0$. If we shift the noise by b along with the initial condition the solution just gets shifted as well. So without loss of generality, we can assume that $b = 0$. Furthermore, if $z(t,x)$ is a solution of the stochastic heat equation (131) then so is $\gamma^{-1} z(t,x)$. So we can also assume $\gamma = 1$. By the comparison principle, Proposition 3.9, we therefore have the following problem: *Let $t > 0$. Under the assumption that $z(0,x) = \mathbf{1}_{[-a,a]}(x)$, show that $P(z(t,x) > 0$ for $|x| \le M) > 1-\delta$ for all $M, \delta > 0$.* To prove this, let

$$\mathscr{A}_k = \left\{z(\tfrac{tk}{\ell}, x) \ge \tfrac{1}{8^k}\mathbf{1}_{[-a-Mk\ell^{-1},a+Mk\ell^{-1}]}(x)\right\}. \tag{164}$$

Suppose we could show that

$$P(\mathscr{A}_{k+1}^C \mid \mathscr{A}_1 \cap \cdots \cap \mathscr{A}_k) < \tfrac{\delta}{\ell}. \tag{165}$$

Then we would have $P(\mathscr{A}_n^C) \le \sum_{k=0}^{n-1} P(\mathscr{A}_{k+1}^C \mid \mathscr{A}_1 \cap \cdots \cap \mathscr{A}_k) < \delta$ as desired.

So we concentrate on (165). First of all note that by the Markov property, it is the same thing as proving $P(\mathscr{A}_{k+1}^C \mid \mathscr{A}_k) < \frac{\delta}{\ell}$. By the comparison principle Proposition 3.9, and the linearity of the stochastic heat equation (131) in the initial data, it is enough to show that (*) *there is an $\ell > 0$ so that if we start with $z(0,x) = \mathbf{1}_{[-a-\frac{Mk}{\ell},a+\frac{Mk}{\ell}]}(x)$, then $z(\frac{t}{\ell},x) \ge \frac{1}{8}\mathbf{1}_{[-a-M(k+1)\ell^{-1},a+M(k+1)\ell^{-1}]}(x)$ with probability at least $1-\delta/\ell$.*

To show (*), note that

$$z(\tfrac{t}{\ell},x) = \int_{a-\frac{Mk}{\ell}}^{a+\frac{Mk}{\ell}]} p(\tfrac{t}{\ell}, x-y)dy + \int_0^{\frac{t}{\ell}}\int_{\mathbb{R}} p(\tfrac{t}{\ell}-s, x-y) z(s,y)\xi(s,y)dyds. \tag{166}$$

The key point is now that for small times, the first, deterministic, term dominates. We are interested in a lower bound for it for x on $[-a-M(k+1)\ell^{-1}, a+M(k+1)\ell^{-1}]$. By inspection, the minimum is at $x = a+M(k+1)\ell^{-1}$, where by normalizing the Gaussian integral,

$$\min_{|x|\le a+M(k+1)\ell^{-1}} \int_{a-\frac{Mk}{\ell}}^{a+\frac{Mk}{\ell}} p(\tfrac{t}{\ell}, x-y)dy = \int_{Mt^{-1/2}\ell^{-1/2}}^{2at^{-1/2}\ell^{1/2}+M(2k+1)t^{-1/2}\ell^{-1/2}} p(1,x)dx > \tfrac{1}{4} \tag{167}$$

for sufficiently large ℓ. Hence what we need to show is that for sufficiently large ℓ, with probability at least $1-\delta/\ell$,

$$\sup_{|x|\le a+M(k+1)\ell^{-1}}\left|\int_0^{\frac{t}{\ell}}\int_{\mathbb{R}} p(\tfrac{t}{\ell}-s,x-y)z(s,y)\xi(s,y)dyds\right|<\tfrac18. \tag{168}$$

The proof will be based on the large deviation estimate Lemma 3.14. But first let's see roughly why it is true. One believes that the supremum of such objects behaves, up to constants, like a typical value. So we drop the sup and compute the variance

$$E\left[\left(\int_0^{\frac{t}{\ell}}\int_{\mathbb{R}} p(\tfrac{t}{\ell}-s,x-y)z(s,y)\xi(s,y)dyds\right)^2\right]=\int_0^{\frac{t}{\ell}}\int_{\mathbb{R}} p^2(\tfrac{t}{\ell}-s,x-y)E[z^2(s,y)]dyds. \tag{169}$$

Recall that $E[z^2(s,y)]\le Cu^2(s,y)$ where u solves the deterministic heat equation with the same initial data. So the variance is bounded by

$$\int_0^{\frac{t}{\ell}}\int_{\mathbb{R}} p^2(\tfrac{t}{\ell}-s,x-y)\left(\int_{|b|\le a+Mk\ell^{-1}} p(s,y-b)db\right)^2 dyds\to 0 \tag{170}$$

as $\ell\to\infty$. In other words, we should be able to choose an ℓ large enough that (168) holds.

To make it rigorous, we can assume without loss of generality that there is a $K<\infty$ such that $z(t,x)<K$ throughout, for by the regularity theorem, $z(t,x)$ is continuous and bounded, so $P(\sup_{\substack{0\le s\le t\\|x|\le r}} z(t,x)>K)\to 0$, so we can choose K large enough that this probability is less than $\delta/2$. Then, by Lemma 3.14, the probability of the complement of (168) is bounded by $C\exp\left\{-C\ell^{1/2}t^{-1/2}K^{-2}\right\}$ which is less than $\delta/2$ for sufficiently large ℓ. $\square$

Remark 3.15. It is worth noting that the proof after (166), comparing the deterministic and stochastic parts, works also for $\partial_t z=\frac12\partial_x^2 z+\xi z^\gamma$ when $\gamma<1$, although we know that in that case, the stochastic part wins and the solution has compact support. The reason the proof does not work is that at each step k, in the reduction to (*), in the $\gamma<1$ case one ends up increasing the noise. If $\gamma>1$, the noise is reduced at each iteration. This is why the linear case is critical for the argument.

3.10.1. *Regularity of the one-point density.* Let $z(t,x)$ be the solution of the stochastic heat equation. We will be studying $F(s)=P(z(t,x)\le s)$. The density (if it exists) is $f(s)=F'(s)$. Using Malliavin calculus it can be shown[6] [PZ93],[MN] that $f(s)$ is a smooth function. The key ingredient is the fact that $z(t,x)>0$.

3.11. **The continuum random polymer.** Fix a space-time white noise $\xi(t,x)$. The (point-to-point) continuum random polymer is the probability measure $P^{\xi}_{0,0,T,x}$ on continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$ and $x(T)=x$ and formal density

$$e^{\int_0^T \xi(t,x(t))dt} \tag{171}$$

with respect to Brownian motion. Unfortunately the density does not make sense. Once again, it needs a type of infinite renormalization. This can be achieved in several ways. We could smooth out the noise, construct the measure with the smoothed out noise, and

[6]The results are for the stochastic heat equation on a finite interval with Dirichlet boundary conditions

then obtain our desired measure as a limit as the smoothing is removed. We will give an alternate construction which is direct and stresses the Markov property.

Let $z(s,x,t,y)$ denote the solution of the stochastic heat equation after time s starting with a delta function at x,

$$\begin{aligned}&\partial_t z = \tfrac{1}{2}\partial_y^2 z + \xi z, \qquad t > s,\ y \in \mathbb{R},\\ &z(s,x,s,y) = \delta_x(y).\end{aligned} \tag{172}$$

It is important that they are all using the same noise ξ.

$P^{\xi}_{0,0,T,x}$ is defined to be the probability measure on continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$ and $x(T)=x$ and finite dimensional distributions

$$\begin{aligned}&P^{\xi}_{0,0,T,x}(x(t_1)\in dx_1,\ldots,x(t_n)\in dx_n) \\ &= \frac{z(0,0,t_1,x_1)z(t_1,x_1,t_2,x_2)\cdots z(t_{n-1},x_{n-1},t_n,x_n)z(t_n,x_n,T,x)}{z(0,0,T,x)}dx_1\cdots dx_n\end{aligned} \tag{173}$$

for $0<t_1<t_2<\cdots<t_n<T$.

One can check these are a consistent family of finite dimensional distributions. It is basically because of the Chapman-Kolmogorov equation

$$\int_{\mathbb{R}} z(s,x,\tau,u)z(\tau,u,t,y)du = z(s,x,t,y) \tag{174}$$

which is a consequence of the uniqueness of solutions of the stochastic heat equation. So there is a probability measure on the product space. One can then show using essentially the Kolmogorov criteria, that the paths are Hölder of any exponent less than $1/2$, just like Brownian motion. In particular, they are continuous. Using (174) the resulting process is Markovian. The quadratic variation can be computed and it is the same as Brownian motion. However, for almost every realization of the noise ξ, $P^{\xi}_{0,0,T,x}$ is singular with respect to the Brownian bridge $P^{0}_{0,0,T,x}$.

One can also define the point-to-line continuum random polymer $P^{\xi}_{0,0}$ on continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. Its finite dimensional distributions are

$$\begin{aligned}&P^{\xi}_{0,0}(x(t_1)\in dx_1,\ldots,x(t_n)\in dx_n, x(t)\in dx) \\ &= \frac{z(0,0,t_1,x_1)z(t_1,x_1,t_2,x_2)\cdots z(t_{n-1},x_{n-1},t_n,x_n)z(t_n,x_n,T,x)}{\int_{\mathbb{R}} z(0,0,T,x)dx}dx_1\cdots dx_n dx\end{aligned} \tag{175}$$

The paths $x(t)$ look like Brownian motions under $P^{\xi}_{0,0}$ on a small scale, but on a large scale (i.e. for large T) they are expected to have the scaling $x(T)\sim T^{2/3}$.

The continuum random polymer is the limit of diffusively rescaled polymer paths in the intermediate scaling regime as long as the random environment satisfies $E[\omega_-^8]<\infty$ [AKQ12a].

3.12. **Universality of KPZ and the continuum random polymer: The weakly asymmetric limit.** There is another choice of scaling in (15). If one takes $\beta=1/2$ and $z=2$, then (13) turns

$$\partial_t h = -\tfrac{1}{2}\epsilon^{1/2}(\partial_x h)^2 + \tfrac{1}{2}\partial_x^2 h + \xi \tag{176}$$

into

$$\partial_t h_\epsilon = -\tfrac{1}{2}(\partial_x h_\epsilon)^2 + \tfrac{1}{2}\partial_x^2 h_\epsilon + \xi. \tag{177}$$

This is the idea behind the weakly asymmetric limit. If a model in the KPZ class has an adjustable asymmetry, then a diffusive scaling of time and space combined with this weak asymmetry should lead to KPZ. KPZ is thus the fixed point of the weakly asymmetric scaling.

3.13. **Intermediate disorder regime for directed random polymers.** We start with the once iterated version of forward equation (69) written in the following way

$$\begin{aligned}&Z(j+2,y)-Z(j,y)=\tfrac{1}{4}\left[Z(j,y+2)-2Z(j,y)+Z(j,y-2)\right]\\&+(e^{-\beta(\xi(j+2,y)+\xi(j+1,y+1))}-1)Z(j,y+2)+(\tfrac{1}{4}e^{-\beta\xi(j+2,y)}(e^{-\beta\xi(j+1,y+1)}+e^{-\beta\xi(j+1,y-1)})-\tfrac{1}{2})Z(j,y)\\&+(e^{-\beta(\xi(j+2,y)+\xi(j+1,y-1))}-1)Z(j,y-2).\end{aligned}$$

Here we drop the initial position from $Z(x,j,y)$ and just write $Z(j,y)$.

The weakly asymmetric limit described above corresponds to look at space scales ϵ^{-1}, time scales ϵ^{-2} and β of order $\epsilon^{1/2}$. So we define

$$\tilde{z}_\epsilon(t,x)=Z_{\beta\epsilon^{1/2}}(0,\lfloor\epsilon^{-2}t\rfloor,\lfloor\epsilon^{-1}x\rfloor) \tag{178}$$

Note that β has changed its meaning. The process $\tilde{z}_\epsilon(t,x)$ lives on the even sites of the lattice $\epsilon^2\mathbb{Z}\times\epsilon\mathbb{Z}$. To keep expressions simple we identify $\tilde{z}_\epsilon(t,x)=\tilde{z}_\epsilon(\epsilon^2\lfloor\epsilon^{-2}t\rfloor,\epsilon[\epsilon^{-1}x])$ where $\lfloor\cdot\rfloor$ and $[\cdot]$ are the greatest and nearest integer functions. Rewriting the discrete forward difference equation,

$$d_\epsilon\tilde{z}_\epsilon(t,x)=\tfrac{1}{2}\Delta_\epsilon\tilde{z}_\epsilon(t,x)+\tilde{\xi}_+(t,x)\tilde{z}_\epsilon(t,x+2\epsilon)+\tfrac{1}{4}(\tilde{\xi}_+(t,x)+\tilde{\xi}_-(t,x))\tilde{z}_\epsilon(t,x)+\tilde{\xi}_-(t,x)\tilde{z}_\epsilon(t,x-2\epsilon) \tag{179}$$

where $d_\epsilon\tilde{z}_\epsilon(t,x):=\frac{1}{2}\epsilon^{-2}(\tilde{z}_\epsilon(t+2\epsilon^2,x)-\tilde{z}_\epsilon(t,x))$, $\Delta_\epsilon\tilde{z}_\epsilon(t,x)=\frac{1}{4}\epsilon^{-2}(\tilde{z}_\epsilon(t,x+2\epsilon)-2\tilde{z}_\epsilon(t,x)+\tilde{z}_\epsilon(t,x-2\epsilon))$ and

$$\tilde{\xi}_+(t,x) = \tfrac{1}{2}\epsilon^{-2}(e^{-\beta\epsilon^{1/2}(\xi(\epsilon^{-2}t+2,\epsilon^{-1}x)+\xi(\epsilon^{-2}t+1,\epsilon^{-1}x+1)}-1) \tag{180}$$

$$\tilde{\xi}_-(t,x) = \tfrac{1}{2}\epsilon^{-2}(e^{-\beta\epsilon^{1/2}(\xi(\epsilon^{-2}t+2,\epsilon^{-1}x)+\xi(\epsilon^{-2}t+1,\epsilon^{-1}x-1)}-1) \tag{181}$$

Recall that $E[\xi(i,j)]=0$. We can write

$$\tilde{\xi}_\pm=\epsilon^{-3/2}\hat{\xi}_\pm+\epsilon^{-1}\gamma_\epsilon \tag{182}$$

where $\hat{\xi}_\pm$ are random variables of mean zero and variance $\mathcal{O}(1)$ (as $\epsilon\to 0$) and $\gamma_\epsilon=\mathcal{O}(1)$ is deterministic. Note that we treat $\beta>0$ as fixed in the computation.

It means we have to subtract a large drift by defining

$$z_\epsilon(t,x)=\epsilon^{-2}(1+\epsilon\tfrac{5}{2}\gamma_\epsilon)^{-\epsilon^{-2}t}\tilde{z}_\epsilon(t,x). \tag{183}$$

Then we can hope that $z_\epsilon(t,x)$ converges to the stochastic heat equation. In particular, $z_\epsilon(t,x)$ satisfies

$$d_\epsilon z_\epsilon(t,x)=\tfrac{1}{2}D_\epsilon\Delta_\epsilon z_\epsilon(t,x)+\epsilon^{-3/2}(\hat{\xi}\cdot z_\epsilon)(t,x) \tag{184}$$

where

$$(\hat{\xi}\cdot z_\epsilon)(t,x):=\hat{\xi}_+(t,x)z_\epsilon(t,x+2\epsilon)+\tfrac{1}{4}(\hat{\xi}_+(t,x)+\hat{\xi}_-(t,x))z_\epsilon(t,x)+\hat{\xi}_-(t,x)z_\epsilon(t,x-2\epsilon) \tag{185}$$

where $D_\epsilon=(1+8\epsilon\gamma_\epsilon)$

Expanding the exponential we have

$$\epsilon^{-2}(e^{-\beta\epsilon^{1/2}\omega}-1)=-\beta\epsilon^{-3/2}\omega+\tfrac{1}{2}\beta^2\epsilon^{-1}\omega^2-\tfrac{1}{3!}\beta^3\epsilon^{-1/2}\omega^3+\tfrac{1}{4!}\beta^4\omega^4+o(1). \tag{186}$$

We write this as

$$\epsilon^{-2}(e^{-\beta\epsilon^{1/2}\omega}-1)=\beta\epsilon^{-3/2}\omega_\epsilon+c_\epsilon \tag{187}$$

where

$$c_\epsilon = \tfrac{1}{2}\beta^2\epsilon^{-1}E[\omega^2] - \tfrac{1}{3!}\beta^3\epsilon^{-1/2}E[\omega^3] + \tfrac{1}{4!}\beta^4 E[\omega^4] + o(1). \tag{188}$$

and

$$\omega_\epsilon = -\omega + \tfrac{1}{2}\beta\epsilon^{1/2}(\omega^2 - E[\omega^2]) - \tfrac{1}{3!}\beta^2\epsilon(\omega^3 - E[\omega^3]) + \tfrac{1}{4!}\epsilon^{3/2}\beta^3(\omega^4 - E[\omega^4]) \tag{189}$$

are new mean zero i.i.d. random variables which are asymptotically the same as the ω's.

If we define

$$z_\epsilon(t,x) = \epsilon^{-1}(1+\epsilon^2 c_\epsilon)^{-\epsilon^{-2}t}\tilde{z}_\epsilon(t,x) \tag{190}$$

then we have

$$N_t^\epsilon(\varphi) = \epsilon\sum_{\epsilon\mathbb{Z}}\varphi(x)z_\epsilon(t,x) - \varphi(0) - \tfrac{1}{2}\epsilon^2\sum_{s\in[0,t]\cap\epsilon^2\mathbb{Z}}\epsilon\sum_{x\in\epsilon\mathbb{Z}} D_\epsilon\Delta_\epsilon\varphi(x)z_\epsilon(s,x) \tag{191}$$

is a martingale at times $t\in[0,t]\cap\epsilon^2\mathbb{Z}$. The first ϵ^{-1} in (190) is to make the initial Dirac mass, i.e. to get the term $\varphi(0)$ to come in unscaled. The $D_\epsilon = 1+\epsilon^2 c_\epsilon$, so $D_\epsilon\to 1$. In fact, $N_t^\epsilon(\varphi)$ is given explicitly as

$$N_t^\epsilon(\varphi) = \epsilon^2\sum_{s\in[0,t]\cap\epsilon^2\mathbb{Z}}\epsilon\sum_{x\in\epsilon\mathbb{Z}}\varphi(x)\bar{z}_\epsilon(s,x)\beta\epsilon^{-3/2}\omega_\epsilon(s+\epsilon^2,x). \tag{192}$$

where $\bar{z}_\epsilon(s,x) := \frac{1}{2}[z_\epsilon(s,x+\epsilon)+z_\epsilon(s,x-\epsilon)]$. Since $\omega_\epsilon(s,x)$, $s\in[0,t]\cap\epsilon^2\mathbb{Z}$, $x\in\epsilon\mathbb{Z}$ are all i.i.d. and in addition, $\omega_\epsilon(s,x)$, $x\in\epsilon\mathbb{Z}$ are independent of $\mathscr{F}_s = \sigma(z(u,x), u\in[0,s]\cap\epsilon^2\mathbb{Z}, x\in\epsilon\mathbb{Z})$,

$$\Lambda_t(\varphi) = N_t(\varphi)^2 - \epsilon^4\sum_{s\in[0,t]\cap\epsilon^2\mathbb{Z}}\epsilon^2\sum_{x\in\epsilon\mathbb{Z}}\varphi^2(x)\bar{z}_\epsilon^2(s,x)\beta^2\epsilon^{-3}E[\omega_\epsilon^2] \tag{193}$$

is a martingale. The ϵ^{-3} is just right and the equation can be rewritten

$$\Lambda_t(\varphi) = N_t(\varphi)^2 - \beta^2\sigma_\epsilon^2\epsilon^2\sum_{s\in[0,t]\cap\epsilon^2\mathbb{Z}}\epsilon\sum_{x\in\epsilon\mathbb{Z}}\varphi^2(x)\bar{z}_\epsilon^2(s,x). \tag{194}$$

From (191) and (194) is it clear that once we show tightness, the limiting process must have (150) and (151) as martingales. The uniqueness of the martingale problem then identifies the limit as the solution of the stochastic heat equation. Before we turn to the tightness, let us make a few remarks and state the general result.

Remark 3.16. The above argument shows that the rescaled unconditioned point-to-point partition function converges to the stochastic heat equation. By unconditioned, we just mean that we take the expectation on the set of paths which end up at a point, instead of conditioning them to end up at that point. To get the conditioning, one just divides by the probability to end of that point, which includes a term of order ϵ, cancelling the ϵ^{-1} in (190). To get the point-to-line partition function asymptotics also requires at most very slight tweaks of the above proof.

Remark 3.17. It would appear that the argument depends on the existence of exponential moments of the random field ω. In fact, one only needs the finiteness of a few moments. Without the intermediate scaling, the free energy is predicted to have the Tracy-Widom asymptotics as soon as $E[\omega_-^5]<\infty$ where $\omega_- = \max(-\omega,0)$. [BBP07]. A similar reasoning leads to the prediction that for the intermediate scaling, $E[\omega_-^6]<\infty$ is enough. Let us explain roughly the argument when $E[\omega_-^8]<\infty$, which is simpler. Checking the proof above, one can see that we get into trouble at (187) when $\omega \ll -\epsilon^{-1/2}$. By Chebyshev's inequality,

$$P(\omega < -\epsilon^{-1/2}) \le \epsilon^4 E[\omega_-^8 \mathbf{1}_{\omega_- < \epsilon^{-1/2}}] = o(\epsilon^4). \tag{195}$$

There are only order ϵ^{-4} sites in our lattice that the random walk path can possibly visit. Hence the probability that even one site has $\omega < -\epsilon^{-1/2}$ goes to zero with ϵ. So we can work with the cutoff variables without affecting the result. To see that $E[\omega_-^6] < \infty$ should really be enough, just note that the random walk paths really only visit ϵ^{-3} sites.

To state the main result, we extend the field $z_\epsilon(t,x)$ of (190) which in principle lives on $\epsilon^2\mathbb{Z}_+ \times \epsilon\mathbb{Z}$, to $\mathbb{R}_+ \times \mathbb{R}$. It doesn't really matter how we do this, but we could first make the paths continuous in x by drawing little straight lines between $z_\epsilon(t,x)$ and $z_\epsilon(t,x+\epsilon)$. Then we could draw straight lines from $z_\epsilon(t,x)$ to $z_\epsilon(t+\epsilon^2,x)$. The result is a continuous function on $\mathbb{R}_+ \times \mathbb{R}$. Let's call its distribution P_ϵ. So for each $\epsilon > 0$ we have a probability measure on $C([0,\infty), C(\mathbb{R}))$ where C means continuous functions with the topology of uniform convergence on compact sets. The main result is

Theorem 3.18 ([AKQ12b]). *Suppose that $E[\omega_-^8] < \infty$. The measures P_ϵ are tight and have a unique weak limit P which is the solution of the stochastic heat equation with initial condition $z(0,x) = \delta_0(x)$.*

As mentioned earlier, we expect the result is true under the weaker condition $E[\omega_-^6] < \infty$. We have sketched above the argument to identify the limit. Below we sketch the argument for the tightness.

First we note an important corollary. Recall the key open problem for directed polymers in one dimension is to show that the free energy has the asymptotics

$$\log E_x\left[e^{-\beta\sum_{i=0}^N \omega(i,x_i)}\right] \sim cN + N^{1/3}\zeta \tag{196}$$

where the fluctuations ζ have the GUE Tracy-Widom distribution. It is supposed to be true for any distribution for the i.i.d. ω satisfying $E[\omega_-^5] < \infty$, though not a single example is known. Combining Theorem 3.18 and Theorem 2.2 we have

Corollary 3.19 (Weak universality). *Suppose that $E[\omega_-^8] < \infty$. Then*

$$\lim_{\beta\to\infty}\lim_{N\to\infty} \frac{\log E_x\left[e^{-\frac{\beta}{N^{1/4}}\sum_{i=0}^N \omega(i,x_i)}\right] - \frac{1}{2}\beta^2 N^{1/2}}{\beta^{4/3}} \stackrel{\text{dist}}{=} F_{GUE} \tag{197}$$

3.14. **Tightness of the approximating partition functions.** We just give a sketch of the tightness. It comes down to obtaining some modulus of continuity estimates which hold uniformly in $\epsilon > 0$. Start with the equation for (190) which we can write as

$$z_\epsilon(t,x) = \epsilon\sum_{y\in\epsilon\mathbb{Z}} p_\epsilon(t-\delta,x-y)z_\epsilon(\delta,y) + \epsilon^2 \sum_{s\in[\delta,t]\cap\epsilon^2\mathbb{Z}} \epsilon\sum_{y\in\epsilon\mathbb{Z}} p_\epsilon(t-s,x-y)z_\epsilon(s,y)\omega_\epsilon(s,y) \tag{198}$$

for any $\delta \in [0,t)$. Here $p_\epsilon(t,x)$ are the discrete heat kernels, i.e. the transition densities of the underlying random walks, normalized so that as ϵ tends to zero, $p_\epsilon(t,x)$ tends to the standard Brownian transition probability density $p(t,x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}$. In exactly the same way as in (143) we get the apriori bound

$$E\left[z_\epsilon^2(t,y)\right] \le Cp_\epsilon^2(t,y). \tag{199}$$

By Burkholder's inequality we also have

$$E\left[z_\epsilon^M(s,y)\right] \le C_M p_\epsilon^M(s,y). \tag{200}$$

From (199) we have good modulus of continuity for the first term on the right side of (198). Now we try to develop them for the second term which we write as

$$U_{\epsilon,\delta}(x,t) = \int_\delta^t \int p_\epsilon(t-s,x-y) z_\epsilon(s,y)\omega_\epsilon(s,y) \tag{201}$$

introducing the integral notation for the sums to keep away from huge expressions. Anyway, if we extend these functions from the lattice to $\mathbb{R}_+ \times \mathbb{R}$, the sums really are replaced by integrals. By Burkholder we can estimate $E[|U_\epsilon(x+\gamma,t) - U_\epsilon(x,t)|^M]$ by

$$C_M E\left[\left(\int_\delta^t \int (p_\epsilon(t-s,x+\gamma-y) - p_\epsilon(t-s,x-y))^2 z_\epsilon^2(s,y)\right)^{M/2}\right] \tag{202}$$

Apply Hölder with $p = M/2, q = M/(M-2)$ to bound this by

$$C'_M E\left[\int_\delta^t \int z_\epsilon^2(s,y)\right]\left[\int_\delta^t \int (p_n(t-s,x+\gamma-y) - p_n(t-s,x-y))^{\frac{2M}{M-2}}\right]^{\frac{M-2}{2}}$$

From (199) we can bound on the first term independent of ϵ (it does depend on t, δ). And one can check that there is a C also depending only on $t, \epsilon > 0$ such that

$$\left[\int_\delta^t \int (p_\epsilon(t-s,x+\gamma-y) - p_\epsilon(t-s,x-y))^{\frac{2M}{M-2}}\right]^{\frac{M-2}{2}} \le C\gamma^{\frac{M}{2}-1}.$$

Now we try to do the same thing for $E[|U_\epsilon(x,t+h) - U_\epsilon(x,t)|^M]^{1/M}$. We estimate it by constant multiples of two terms

$$E\left[\left|\int_\delta^t \int (p_\epsilon(t+h-s,x-y) - p_\epsilon(t-s,x-y))^2 z_\epsilon^2(s,y)\right|^{\frac{M}{2}}\right]^{\frac{1}{M}} \tag{203}$$

and

$$E\left[\left|\int_t^{t+h} \int p_\epsilon^2(t+h-s,x-y) z_\epsilon^2(s,y)\right|^{\frac{M}{2}}\right]^{\frac{1}{M}} \tag{204}$$

The first one we estimate by Hölder with $p = M/2, q = M/(M-2)$,

$$E\left[\int_\delta^t \int z_\epsilon^M(s,y)\right]^{\frac{1}{M}} \left(\int_\delta^t \int |p_\epsilon(t+h-s,x-y) - p_\epsilon(t-s,x-y)|^{\frac{2M}{M-2}}\right)^{\frac{M-2}{2M}}$$

The apriori bound (198) controls the expectation. The second term is bounded in ϵ when $M > 8$, in which case it can be computed and is less than $C_M h^{\frac{1}{4}-\frac{2}{M}}$. (204) we estimate by Hölder again

$$E\left[\int_t^{t+h} \int z_\epsilon^M(s,y)\right]^{\frac{1}{M}} \left(\int_\delta^t \int p_\epsilon^2(t-s,x-y)\right)^{\frac{M-2}{2M}} \tag{205}$$

From the apriori bound this is again less than $C_M h^{\frac{1}{4}-\frac{2}{M}}$.

We obtain

Lemma 3.20. *For each even $M > 8$, and each $\delta > 0$, there is a $C_M < \infty$ such that for $t, t+h \geq \delta$,*

$$E[|z_n(x+\gamma, t+h) - z_n(x,t)|^M]^{1/M} \leq C_M \max(\gamma, h)^{\frac{1}{4}-\frac{2}{M}}. \tag{206}$$

Note that if we were more careful we could improve the modulus of continuity to Hölder $1/2-$ in space, but for tightness we do not need an optimal result.

Now we use the inequality of Garsia [Gar72] that

$$|f(x) - f(y)| \leq 8 \int_0^{|x-y|} \Psi^{-1}(B/u^{2d}) dp(u) \tag{207}$$

for all functions f continuous in a unit cube $I \subset \mathbf{R}^d$ that satisfy the inequality

$$\int_I \int_I \Psi(f(x) - f(y)/p(d^{-1/2}|x-y|))\, dxdy \leq B$$

for all hypercubes $I \subset I_0$, where (i) $e(I)$ denotes the volume of I, (ii) Ψ is non-constant even convex function and (iii) p is a continuous even function that satisfies the condition $\lim_{n\to 0} p(u) = 0$.

We are working in $d = 2$ (space+time). Choosing $\Psi(x) = x^M$, $M > 6$ and $p(x) = x^{\gamma/M}$ we have from Lemma 3.20.

$$E\left[\int_{t,s\in[\delta,T],x,y\in\mathbb{R}} \Psi\left(\frac{|z_n(x,t) - z_n(y,s)|}{p(2^{-1/2}\sqrt{(t-s)^2 + (x-y)^2})}\right)\right] \leq C_M \tag{208}$$

Here we have extended $z_n(t,x)$ as a continuous function to $[0,T] \times \mathbb{R}$.

Since $\int_0^h \Psi^{-1}(B/u^{2d})dp(u) = C_{M,\gamma} B^{1/M} h^{\frac{\gamma-4}{M}}$ with a finite $C_{M,\gamma}$ for $\gamma > 4$, we conclude that if $\mathcal{H}_{[\delta,T]\times\mathbb{R}}(\alpha, K)$ denotes the set of functions $z(t,x)$ on $[\delta, T] \times \mathbb{R}$ with

$$|z(t,x) - z(s,y)| \leq K|(t-s)^2 + (x-y)^2|^{\alpha/2},$$

then

Lemma 3.21. *If P_n denotes the distribution of $z_n(t,x)$ then for any $\delta > 0$ and and $\alpha < 1/4$,*

$$\limsup_{K\to\infty} \limsup_{n\to\infty} P_n(\mathcal{H}_{[\epsilon,T]\times\mathbb{R}}(\alpha, K)) = 1. \tag{209}$$

In particular, since $\mathcal{H}_{[\epsilon,T]\times\mathbb{R}}(\alpha, K)$ are compact sets of $C([\delta, T] \times \mathbb{R})$, the P_n are tight.

3.15. **Asymmetric exclusion.** The asymmetric simple exclusion process (ASEP) with parameters $p, q \geq 0$, $p + q = 1$,$p \neq q$ is a continuous time Markov process on $S = \{0,1\}^{\mathbb{Z}}$, the 1's being thought of as particles and the 0's as holes. Each particle waits a random exponent mean one amount of time and then attempts a jump, one site to the right with probability p and one site to the left with probability q. However, the jump is only performed if there is no particle at the target site. Otherwise, nothing happens and the particle waits another exponential time. All particles are doing this independently of each other.

The generator of ASEP acts on local functions (functions which depend on only finitely many coordinates) by

$$Lf(\eta) = \sum_x \{p\eta(x)(1-\eta(x+1)) + q(1-\eta(x))(\eta(x+1))\} (f(\eta^{x,x+1}) - f(\eta)) \tag{210}$$

where $\eta^{x,x+1} \in \{0,1\}^{\mathbb{Z}}$ is obtained from η by switching the occupation variables at x and $x+1$. The corresponding Markov semigroup acts on the Banach space $C(S)$ of continuous functions on S with sup norm $\|f\| = \sup_{\eta\in S} |f(\eta)|$ by

$$P_t f(\eta) = E[f(\eta(t)) \mid \eta(0) = \eta]. \tag{211}$$

The domain of the generator L is given by

$$\mathcal{D} = \left\{ f \in C(S) : Lf = \lim_{t\searrow 0} t^{-1}(P_t f - f) \text{ exists} \right\}.$$

The problem with such a definition of course is that it is very hard to tell which f are in $\mathcal{D}$. So the following definition is important.

Definition 3.22. A subspace $\mathcal{D}_0 \subset \mathcal{D}$ is a *core* for L if L is the closure of its restriction to $\mathcal{D}_0$.

In particular, one can check invariance of a measure on the core:

Lemma 3.23 (Prop. I.2.13 of [Lig85])**.** *If $\mathcal{D}_0$ is a core for L, then a probability measure μ is invariant for the semigroup P_t, i.e. $P_t^*\mu = \mu$ if and only if $\int Lf d\mu = 0$ for all $f \in \mathcal{D}_0$.*

A function on the state space $\{0,1\}^{\mathbb{Z}}$ is called *local* if it only depends on finitely many coordinates. For ASEP we have the important

Lemma 3.24 (Thm. I.3.9 of [Lig85])**.** *The subspace $\mathcal{D}_0$ of local functions is a core for the generator (210).*

Hence we can check the following

Proposition 3.25. *For any $\rho \in [0,1]$ the Bernoulli product measure π_ρ on $\{0,1\}^{\mathbb{Z}}$ with $\pi_\rho(\eta(x) = 1) = \rho$ and $\pi_\rho(\eta(x) = 0) = 1 - \rho$ is invariant for ASEP.*

Proof. Let f be a local function, depending on $\eta(x)$, $|x| \le B$. We want to show $\int Lf d\pi_\rho = 0$. We can assume $\int f d\pi_\rho = 0$, for otherwise we can subtract a constant to make it so, without affecting Lf. Changing η to $\eta^{x,x+1}$ we have

$$\begin{aligned}\int \eta(x)(1-\eta(x+1)) f(\eta^{x,x+1}) d\pi_\rho(\eta) &= \int \eta(x+1)(1-\eta(x)) f(\eta) \frac{d\pi_\rho(\eta^{x,x+1})}{d\pi_\rho(\eta)} d\pi_\rho(\eta) \\ &= \int \eta(x+1)(1-\eta(x)) f(\eta) d\pi_\rho(\eta)\end{aligned}$$

since $\frac{d\pi_\rho(\eta^{x,x+1})}{d\pi_\rho(\eta)} = 1$. Hence

$$\int Lf d\pi_\rho = \int (p-q) \sum_x (\eta(x+1)(1-\eta(x) - \eta(x)(1-\eta(x+1)) f(\eta) d\pi(\eta). \tag{212}$$

Note that the summation is telescoping. Hence it can be rewritten as $\int gf d\pi_\rho$ where g does not depend on the variables $\eta(x)$, $|x| \le B$. Since π_ρ is a product measure $\int gf d\pi_\rho = \int g d\pi_\rho \int f d\pi_\rho$ which vanishes by assumption. □

The π_ρ are the extremals of the set of translation invariant probability measures invariant for ASEP. There are other invariant measures which are not translation invariant, e.g. the *blocking measures* which are product measures with $\mu(\eta(x) = 1) = \frac{(p/q)^x}{1+(p/q)^x}$.

3.15.1. *Height function.* Define $\hat\eta(x) = 2\eta(x) - 1$ which take values $\{-1, 1\}$ instead of $\{0, 1\}$ and define the height function of ASEP by

$$h(t,x) = \begin{cases} 2N(t) + \sum_{0<y\le x} \hat\eta(t,y), & x > 0, \\ 2N(t), & x = 0, \\ 2N(t) - \sum_{x<y\le 0} \hat\eta(t,y), & x < 0, \end{cases} \tag{213}$$

where $N(t)$ is the net number of particles which crossed from site 1 to 0 up to time t., meaning that particle jumps $1 \to 0$ are counted as $+1$ and jumps $0 \to 1$ are counted as -1. We usually think of $h(x)$ in terms of its linear interpolation on $\mathbb{R}$. The reason for the funny definition with the $N(t)$ is that, defined this way, the $h(t,x)$ is the Markov process with state space simple random walk paths with the very simple dynamics that local minima jump to local maxima at rate q and local maxima jump to local minima at rate p.

3.16. **Weakly asymmetric limit of simple exclusion.** We will consider the weakly asymmetric simple exclusion with

$$q - p = \epsilon^{1/2} \tag{214}$$

We have to choose some parameters carefully to make the proof work. Then we will explain them. Let

$$\nu_\epsilon = p + q - 2\sqrt{qp} = \tfrac{1}{2}\epsilon + \tfrac{1}{8}\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{215}$$

and

$$\lambda_\epsilon = \tfrac{1}{2}\log(q/p) = \epsilon^{1/2} + \tfrac{1}{3}\epsilon^{3/2} + \mathcal{O}(\epsilon^{5/2}). \tag{216}$$

Recall the definition (213) of the height function. The rescaled height function is

$$h_\epsilon(t,x) := \lambda_\epsilon h(\epsilon^{-2}t, \epsilon^{-1}x) + \epsilon^{-2}\nu_\epsilon t \tag{217}$$

The main result is that $h_\epsilon(t,x)$ converge to the Hopf-Cole solution of KPZ. What this means of course is that

$$z_\epsilon(t,x) = e^{-h_\epsilon(t,x)} \tag{218}$$

converges to the solution of the stochastic heat equation. This was first proved by Bertini and Giacomin [BG97] in the case of data close to equilibrium, and extended to the case of step initial data (Dirac initial data for the stochastic heat equation) in [ACQ11]. For step initial data one has to multiply $z_\epsilon(t,x)$ by a large factor $\epsilon^{-1/2}$ in order to get the initial Dirac in the limit. Since the stochastic heat equation is linear, one has a fair amount of freedom to do such things.

We now state the main results. $z_\epsilon(t,x)$ really lives on $\mathbb{R}_+ \times \epsilon\mathbb{Z}$, but we can extend it easily to $\mathbb{R}_+ \times \mathbb{R}$ by putting in little diagonal lines. We still call it $z_\epsilon(t,x)$. It is not continuous in t, but the jumps are small. Usually, convergence of discontinuous stochastic processes in done on the Skorohod space D of right continuous paths with left limits. But our jumps are small, and vanish in the limit, so the Skorohod topology is unnecessary. We can just use the uniform topology on D. We call the resulting space D_u. So our process lives in $D_u([0,\infty); C(\mathbb{R}))$. We have such a process $z_\epsilon(t,x)$ for each $\epsilon > 0$ and its distribution is a probability measure P_ϵ on $D_u([0,\infty); C(\mathbb{R}))$. The initial distibutions μ_ϵ, $\epsilon \in (0, 1/4)$ are the distributions of $h_\epsilon(0,x)$ or $z_\epsilon(0,x)$ under P_ϵ.

Theorem 3.26 ([BG97]). *Suppose that the initial distribution μ_ϵ satisfies*

(1) *μ_ϵ converge weakly to a limit μ supported on continuous functions.*

(2) *For each* $n = 1, 2, \dots$ *there is an* $a = a_n$ *and* $c = c_n$ *such that for all* $x \in \mathbb{R}$.

$$E_{\mu_\epsilon}[z^n(0,x)] \leq ce^{a|x|} \tag{219}$$

and

$$E_{\mu_\epsilon}[|h(0,x) - h(0,y)|^{2n}] \leq ce^{a(|x|+|y|)}(|x-y|). \tag{220}$$

Then P_ϵ*, for* $\epsilon \in (0, 1/4)$*, are a tight family of measures and the unique limit point is supported on* $C((0,\infty); C(\mathbb{R}))$ *(continuous in both space and time) and corresponds to the solution of the SHE with initial data* $z(0,x)$ *distributed as* μ.

Taking initial data Bernoulli 1/2 we have a stationary process $u_\epsilon(t,x) = \epsilon^{-1}(h_\epsilon(t,x+\epsilon) - h_\epsilon(t,x))$ for each $\epsilon \in (0,1/4)$. The convergence of the h_ϵ is interpreted as distributional convergence of the u_ϵ. The limiting process $u(t,x)$ means nothing other than the distributional derivative in x of the limiting process $h(t,x)$. Since it is the limit of stationary processes, it is stationary itself. In this way, [BG97] prove that white noise is stationary for the stochastic Burgers equation. Remarkably, this is the only proof known.

The case of step initial data is not included in Theorem 3.26. Here μ_ϵ is concentrated on the special configuration with all zero's to the left of the origin and all one's to the right. The initial height function is $|x|$. So $h_\epsilon(0,x) = \lambda_\epsilon|\epsilon^{-1}x| \sim \epsilon^{-1/2}|x|$. Hence $z_\epsilon(t,x)$ has to be redefined as

$$z_\epsilon(t,x) = \epsilon^{-1/2}e^{-h_\epsilon(t,x)}. \tag{221}$$

We then have for the corresponding P_ϵ,

Theorem 3.27 ([ACQ11])**.** P_ϵ*,* $\epsilon \in (0,1/4)$*, are a tight family of measures and the unique limit point* P *is supported on* $C((0,\infty); C(\mathbb{R}))$ *and corresponds to the solution of the SHE with initial data* $z(0,x) = \delta_0(x)$.

The proof in [ACQ11] reduces the problem to the setup in [BG97] after a short time. Unfortunately, the proof of the key bound (3.28) in [BG97] is very hard to follow. We will concentrate here on providing a simple straightforward proof, and to do this we will work on a circle geometry $\epsilon\mathbb{Z}/\mathbb{Z}$. We always choose $\epsilon^{-1} \in \mathbb{Z}$ and we fix the number of particles to be $\epsilon^{-1}/2$ so that the height function is properly periodic.

3.16.1. *Gärtner's microscopic Hopf-Cole transformation.* We write the stochastic differential equation which governs the evolution of $z_\epsilon(t,x)$.

$$dz_\epsilon = \Omega_\epsilon z_\epsilon dt + (e^{-2\lambda_\epsilon} - 1)z_\epsilon dM_\epsilon^- + (e^{2\lambda_\epsilon} - 1)z_\epsilon dM_\epsilon^- \tag{222}$$

where

$$\Omega = \epsilon^{-2}\nu + (e^{-2\lambda} - 1)r_- + (e^{2\lambda} - 1)r_+ \tag{223}$$

and $dM_\pm(t,x) = dP_\pm(t,x) - r_\pm(x)dt$ where $P_-(t,x), P_+(t,x),\ x \in \epsilon\mathbb{Z}$ are independent Poisson processes running at rates $r_-(t,x), r_+(t,x)$.

$$r_-(x) = \epsilon^{-2}q(1-\eta(x))\eta(x+1) = \frac{1}{4}\epsilon^{-2}q(1-\hat\eta(x))(1+\hat\eta(x+1)) \tag{224}$$

$$r_+(x) = \epsilon^{-2}p\eta(x)(1-\eta(x+1)) = \frac{1}{4}\epsilon^{-2}p(1+\hat\eta(x))(1-\hat\eta(x+1)) \tag{225}$$

This is just a way of writing that the process jumps at rates

$$r_-(x) \tag{226}$$

to $e^{-2\lambda_\epsilon}z_\epsilon$ and

$$r_+(x) \tag{227}$$

to $e^{2\lambda_\epsilon} z_\epsilon$, independently at each site $x \in \epsilon\mathbb{Z}$.

Let

$$D_\epsilon = 2\sqrt{pq} = 1 - \frac{1}{2}\epsilon + \mathcal{O}(\epsilon^2) \tag{228}$$

and $\Delta_\epsilon f(x) = \epsilon^{-2}(f(x+\epsilon) - 2f(x) + f(x-\epsilon))$ be the $\epsilon\mathbb{Z}$ Laplacian. We also have

$$\tfrac{1}{2} D_\epsilon \Delta_\epsilon z_\epsilon = \tfrac{1}{2}\epsilon^{-2} D_\epsilon (e^{-\lambda_\epsilon \hat\eta(x+1)} - 2 + e^{\lambda_\epsilon \hat\eta(x)}) z_\epsilon \tag{229}$$

The key point is that parameters can be chosen so that

$$\Omega_\epsilon = \tfrac{1}{2}\epsilon^{-2} D_\epsilon (e^{-\lambda_\epsilon \hat\eta(x+1)} - 2 + e^{\lambda_\epsilon \hat\eta(x)}) \tag{230}$$

We can do this because the four cases, corresponding to the four possibilities for $\hat\eta(x), \hat\eta(x+1)$: 11, $(-1)(-1)$, $1(-1)$, $(-1)1$, give four equations in three unknowns,

$$\begin{array}{cl}
11 & \tfrac{1}{2}\epsilon^{-2} D_\epsilon (e^{-\lambda_\epsilon} - 2 + e^{\lambda_\epsilon}) = \epsilon^{-2}\nu_\epsilon \\
(-1)(-1) & \tfrac{1}{2}\epsilon^{-2} D_\epsilon (e^{\lambda_\epsilon} - 2 + e^{-\lambda_\epsilon}) = \epsilon^{-2}\nu_\epsilon \\
1(-1) & \tfrac{1}{2}\epsilon^{-2} D_\epsilon (e^{\lambda_\epsilon} - 2 + e^{\lambda_\epsilon}) = \epsilon^{-2}\nu_\epsilon + (e^{2\lambda_\epsilon} - 1)\epsilon^{-2} p \\
(-1)1 & \tfrac{1}{2}\epsilon^{-2} D_\epsilon (e^{-\lambda_\epsilon} - 2 + e^{-\lambda_\epsilon}) = \epsilon^{-2}\nu_\epsilon + (e^{-2\lambda_\epsilon} - 1)\epsilon^{-2} q
\end{array}$$

Luckily, the first two equations are the same, so it is actually three equations in three unknowns, with solution given by (216), (215) and (228).

Hence [Gär88],[BG97],

$$dz_\epsilon = \tfrac{1}{2} D_\epsilon \Delta_\epsilon z_\epsilon + z_\epsilon dM_\epsilon \tag{231}$$

where

$$dM_\epsilon(x) = (e^{-2\lambda_\epsilon} - 1) dM_-(x) + (e^{2\lambda_\epsilon} - 1) dM_+(x) \tag{232}$$

are martingales in t with

$$d\langle M_\epsilon(x), M_\epsilon(y)\rangle = \epsilon^{-1}\mathbf{1}(x=y) b_\epsilon(\tau_{-[\epsilon^{-1}x]}\eta) dt \tag{233}$$

where $\tau_x \eta(y) = \eta(y-x)$ and

$$b_\epsilon(\eta) = 1 - \hat\eta(1)\hat\eta(0) + \hat b_\epsilon(\eta) \tag{234}$$

where

$$\begin{aligned}
\hat b_\epsilon(\eta) \;=\; & \epsilon^{-1}\{[p((e^{-2\lambda_\epsilon} - 1)^2 - 4\epsilon) + q((e^{2\lambda_\epsilon} - 1)^2 - 4\epsilon)] \\
& + [q(e^{-2\lambda_\epsilon} - 1)^2 - p(e^{2\lambda_\epsilon} - 1)^2](\hat\eta(1) - \hat\eta(0)) \\
& - [q(e^{-2\lambda_\epsilon} - 1)^2 + p(e^{2\lambda_\epsilon} - 1)^2 - \epsilon]\hat\eta(1)\hat\eta(0)\}.
\end{aligned} \tag{235}$$

Clearly $b_\epsilon \geq 0$. It is easy to check that there is a $C < \infty$ such that

$$0 \leq \hat b_\epsilon \leq C\epsilon^{1/2} \tag{236}$$

and, for sufficiently small $\epsilon > 0$,

$$0 \leq b_\epsilon \leq 3. \tag{237}$$

We have the following bound on the initial data. For each $p = 1, 2, \ldots$ there exists $C_p < \infty$ such that for all $x \in \mathbb{R}$,

$$E[z_\epsilon^p(0,x)] \leq e^{C_p|x|}. \tag{238}$$

For any $\delta > 0$ let $\mathscr{P}_\epsilon^\delta$ denote the distribution of $z_\epsilon(t,x)$, $t \in [\delta, \infty)$, as measure on $D[[\delta,\infty), C(\mathbb{R})]$ where D means the right continuous paths with left limits, with the topology of uniform convergence on compact sets. In [BG97], Section 4 it is shown that if (238) holds, then, for any $\delta > 0$, $\mathscr{P}_\epsilon^\delta$, $\epsilon > 0$ are tight. The limit points are consistent as $\delta \searrow 0$, and from the integral version of (231),

$$\begin{aligned} z_\epsilon(t,x) &= \epsilon \sum_{y\in\epsilon\mathbb{Z}} p_\epsilon(t,x-y) z_\epsilon(0,y) \\ &\quad + \int_0^t \epsilon \sum_{y\in\epsilon\mathbb{Z}} p_\epsilon(t-s,x-y) z_\epsilon(s,y) dM_\epsilon(s,y) \end{aligned} \tag{239}$$

where $p_\epsilon(t,x)$ are the transition probabilities for the continuous time random walk with generator $\frac{1}{2}D\Delta_\epsilon$, normalized so that $p_\epsilon(t,x) \to p(t,x) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}$, we have

$$\lim_{t\to 0}\lim_{\epsilon\to 0} E\left[\left(z_\epsilon(t,x) - \epsilon \sum_{y\in\epsilon\mathbb{Z}} p_\epsilon(t,x-y) z_\epsilon(0,y)\right)^2\right] = 0 \tag{240}$$

so that the initial data hold under the limit $\mathscr{P}$. Finally, we need to identify the limit of the martingale terms. Let φ be a smooth test function on $\mathbb{R}$. We hope to show that under $\mathscr{P}$, (150) and (151) are local martingales. We have that

$$N_{t,\epsilon}(\varphi) := \epsilon \sum_{x\in\epsilon\mathbb{Z}} \varphi(x) z_\epsilon(t,x) - \tfrac{1}{2} \int_0^t \epsilon \sum_{x\in\epsilon\mathbb{Z}} D_\epsilon \Delta_\epsilon \varphi(x) z_\epsilon(s,x) ds \tag{241}$$

is a martingale under P_ϵ. So the key point is to identify the quadratic process. We have that

$$\Lambda_{t,\epsilon}(\varphi) := N_{t,\epsilon}(\varphi)^2 - \tfrac{1}{2} \int_0^t \epsilon \sum_{\epsilon\mathbb{Z}} \varphi^2(x) b_\epsilon(s,x) z_\epsilon^2(s,x) ds \tag{242}$$

is a martingale under P_ϵ.

The problem with (242) is that we have the b_ϵ instead of 1. First let's note that even if we don't know what the asymptotic behaviour of b_ϵ is, we do know $|b_\epsilon| \le 3$. Using this, we can recover all the estimates we had before about modulus of continuity. In particular, we have the tightness. But we still need to identify the limit martingales. Let

$$a_\epsilon(t,x) = \hat\eta(\epsilon^{-2}t, x+1)\hat\eta(\epsilon^{-2}t, x) z_\epsilon^2(t,x) \tag{243}$$

and recall that $b_\epsilon z_\epsilon^2 = z_\epsilon^2 + a_\epsilon$.

So the main estimate needed to identify the limit with martingales is

Proposition 3.28 (Bertini-Giacomin [BG97]). *Let*

$$a_\epsilon(t,x) = \hat\eta(\epsilon^{-2}t, x+1)\hat\eta(\epsilon^{-2}t, x) z_\epsilon^2(t,x) \tag{244}$$

Then for any smooth test function $\varphi \in C_0^\infty(\mathbb{R})$ under the weakly asymmetric exclusion P_ϵ,

$$\int_0^t \epsilon \sum_{x\in\epsilon\mathbb{Z}} \varphi^2(x) a_\epsilon(s,x) ds \to 0 \tag{245}$$

Note that $a_\epsilon(t,x)$ is of order 1. The proposition is a kind of ergodic theorem. We expect essentially product measure with density 1/2 under which $\hat\eta(x+1)$ and $\hat\eta(x)$ are independent and mean zero. The term $z_\epsilon^2(t,x)$ shouldn't mess things up too much because we could replace it by $z_\epsilon^2(t,x-\epsilon)$. The error would be of order $\epsilon^{1/2}$, so it doesn't matter. Then, in equilibrium, $z_\epsilon^2(t,x-\epsilon)$ is independent of $\hat\eta(x+1)$ and $\hat\eta(x)$.

Proof. We will give a different proof from the one of Bertini and Giacomin, which comes from hydrodynamic limits. Let μ_ϵ be the probability measure on $\{0,1\}^{\epsilon\mathbb{Z}/\mathbb{Z}}$ corresponding to our random initial data. Let ν be the uniform measure on configurations with $\epsilon^{-1}/2$ particles. Because we are on $\epsilon\mathbb{Z}/\mathbb{Z}$ there are ϵ^{-1} sites and

$$H(\mu_\epsilon/\nu) := E_{\mu_\epsilon}[\log(\mu_\epsilon/\nu)] \le C\epsilon^{-1}. \tag{246}$$

Let E_ϵ denote the expectation with respect to the process starting from μ_ϵ and E_ϵ^{eq} denote the expectation with respect to the equilibrium process starting with π. We have the entropy inequality,

$$E_\epsilon[f_\epsilon] \le K^{-1}\epsilon \log E_\epsilon^{\mathrm{eq}}[e^{K\epsilon^{-1}f_\epsilon}] + K^{-1}\epsilon H(\mu_\epsilon/\nu). \tag{247}$$

Hence if for any $K \in \mathbb{R}$,

$$\limsup_{\epsilon\to 0} \epsilon \log E_\epsilon^{\mathrm{eq}}[e^{K\epsilon^{-1}f_\epsilon}] = 0 \tag{248}$$

we have $E_\epsilon[f_\epsilon] \to 0$.

Our f will be of the form $\epsilon\int_0^s V(\eta(s))ds$. So we have to estimate $\epsilon\log\int u(t,\eta)d\pi_{1/2}(\eta)$ where

$$u(t,\eta) = E_\epsilon\left[\exp\left\{\int_0^t V(\eta(s))ds\right\} \mid \eta(0)=\eta\right]. \tag{249}$$

By Jensen's inequality,

$$\log\int u d\nu \le \tfrac{1}{2}\log\int u^2 d\nu. \tag{250}$$

By the Feynman-Kac formula u solves

$$\partial_t u = Lu + Vu. \tag{251}$$

Multiplying by u and integrating by parts we get

$$\tfrac{1}{2t}\log\int u^2 d\nu = \frac{\frac{1}{t}\int_0^t\left\{\int Vu^2 d\nu - \mathscr{D}(u)\right\}ds}{\int u^2 d\nu} \tag{252}$$

where $\mathscr{D}(u)$ is the Dirichlet form $\epsilon^{-2}\frac{1}{2}\sum_x\int|\nabla_{x,x+1}u|^2 d\nu$ with $\nabla_{x,x+1}u(\eta) = u(\eta^{x,x+1}) - u(\eta)$, $\eta^{x,x+1}$ being the configuration with the occupation variables exchanged between sites x and $x+1$. By the Rayleigh-Ritz formula the last ratio is bounded above by the principle eigenvalue of $S+V$ where $S = L + L^*$ is the generator of the symmetric simple exclusion process (i.e. $p=q$). Thus the problem (248) is reduced to large deviations for a reversible process.

The following estimate combines the one and two block estimates of hydrodynamic limits. Let f be a local function, and let $\bar{f}(\rho) = E_{\pi_\rho}[f]$, where π_ρ is a the product measure with density ρ. Let

$$V_{\epsilon,\delta} = \sum_{x\in\epsilon\mathbb{Z}}\varphi^2(x)\left[f(\tau_x\eta) - \bar{f}(\bar{\eta}_{x,\epsilon^{-1}\delta})\right] \tag{253}$$

where $\bar{\eta}_{x,\epsilon^{-1}\delta}$ is the empirical density in an interval of length $e^{-1}\delta$ around x. Then

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\sup_{\|u\|_2=1}\left\{\int V_{\epsilon,\delta}u^2 d\nu - \mathscr{D}_\epsilon(u)\right\} = 0 \tag{254}$$

We will be using this with

$$f(\eta) = \hat{\eta}(1)\hat{\eta}(0) \tag{255}$$

for which it is easy to see that $\bar{f}(\rho) = (\rho - \frac{1}{2})^2$. Now note that we have apriori that $z(t,x)$ is Hölder $\frac{1}{2}-$ in space. So we can have the same estimate (254) if we replace (253) by

$$V_{\epsilon,\delta} = \sum_{x\in\epsilon\mathbb{Z}} \varphi^2(x)\left[f(\tau_x\eta) - \bar{f}(\bar{\eta}_{x,\epsilon^{-1}\delta})\right] z_\delta^2(t,x). \tag{256}$$

where

$$z_\delta^2(t,x) = \min\left\{z^2(t, x + \tfrac{1}{2}\epsilon^{-1}\delta), z^2(t, x + \tfrac{1}{2}\epsilon^{-1}\delta)\right\}. \tag{257}$$

The big conclusion of all this hydrodynamics argument is that if we want to prove that

$$\limsup_{\epsilon\to 0} P_\epsilon\Big(\int_0^t \epsilon \sum_{x\in\epsilon\mathbb{Z}} \varphi^2(x) a_\epsilon(s,x)ds \geq \lambda\Big) = 0 \tag{258}$$

it suffices to prove that

$$\limsup_{\delta\to 0}\limsup_{\epsilon\to 0} P_\epsilon\Big(\int_0^t \epsilon \sum_{x\in\epsilon\mathbb{Z}} \varphi^2(x)(\bar{\eta}_{x,\epsilon^{-1}\delta})^2 z_\delta^2(t,x)ds \geq \lambda\Big) = 0. \tag{259}$$

To prove this, note that

$$\bar{\eta}_{x,\epsilon^{-1}\delta} = \epsilon^{1/2}\delta^{-1}(\log z_\epsilon(x + \tfrac{1}{2}\epsilon^{-1}\delta) - \log z_\epsilon(x - \tfrac{1}{2}\epsilon^{-1}\delta)) \tag{260}$$

so that

$$\epsilon \sum_{x\in\epsilon\mathbb{Z}} \varphi^2(x)(\bar{\eta}_{x,\epsilon^{-1}\delta})^2 z_\delta^2(t,x) \leq 2\delta^{-2}\epsilon^2 \sum_{x\in\epsilon\mathbb{Z}} \varphi_\delta^2(x)(\log z_\epsilon(x))^2 z_\epsilon^2(x) \tag{261}$$

where $\varphi_\delta(x) = (\varphi_\delta^2(x + \frac{1}{2}\epsilon^{-1}\delta) + \varphi_\delta^2(x - \frac{1}{2}\epsilon^{-1}\delta))^{1/2}$. The last term clearly goes to zero under P_ϵ as $\epsilon \to 0$, for any fixed δ (there is an extra ϵ in front.) This completes the proof. $\square$

3.17. **Steepest descent analysis of the Tracy-Widom step Bernoulli formula.** We only give here a heuristic explanation of the proof. We are taking the limit of (93) with

$$\begin{aligned}
&p = \frac{1}{2} - \frac{1}{2}\epsilon^{1/2}, \qquad q = \frac{1}{2} + \frac{1}{2}\epsilon^{1/2}\\
&\gamma = \epsilon^{1/2}, \qquad \tau = \frac{1-\epsilon^{1/2}}{1+\epsilon^{1/2}}\\
&x = \epsilon^{-1}X, \qquad t = \epsilon^{-3/2}T\\
&m = \frac{1}{2}\left[\epsilon^{-1/2}\left(-s + \log(\epsilon^{-1/2}/2) + \frac{X^2}{2T}\right) + \frac{1}{2}t + x\right]\\
&\left\{H_\epsilon(T,X) - \tfrac{X^2}{2T} - \tfrac{T}{24} \geq -s\right\} = \left\{\mathbf{x}(\frac{t}{\gamma}, m) \leq x\right\}
\end{aligned}$$

and contours

$$\Gamma_\eta = \{z : |z| = 1 - \tfrac{1}{2}\epsilon^{1/2}\} \qquad \text{and} \qquad \Gamma_\zeta = \{z : |z| = 1 + \tfrac{1}{2}\epsilon^{1/2}\}$$

The first term in the integrand of (102) is the infinite product $\prod_{k=0}^\infty (1 - \mu\tau^k)$. Observe that $\tau = q/p \approx 1 - 2\epsilon^{1/2}$ and that S_{τ^+}, the contour on which μ lies, is a circle centered at zero of radius between τ and 1. The infinite product is not well behaved along most of this contour, however we can deform the contour to one along which the product is not highly oscillatory. However, the Fredholm determinant has poles at every $\mu = \tau^k$ and the deformation must avoid passing through them. Observe that if

$$\mu = \epsilon^{1/2}\tilde{\mu} \tag{262}$$

then

$$\prod_{k=0}^{\infty}(1-\mu\tau^k) \approx e^{-\sum_{k=0}^{\infty}\mu\tau^k} = e^{-\mu/(1-\tau)} \approx e^{-\tilde{\mu}/2}.$$

We make the $\mu \mapsto \epsilon^{-1/2}\tilde{\mu}$ change of variables and find that if we consider a $\tilde{\mu}$ contour

$$\tilde{\mathcal{C}}_\epsilon = \{e^{i\theta}\}_{\pi/2\leq\theta\leq3\pi/2} \cup \{x\pm i\}_{0<x\leq\epsilon^{-1/2}-1} \cup \{\epsilon^{-1/2}-1+iy\}_{-1<y<1},$$

then the above approximations are reasonable. Thus the infinite product goes to $\exp\{-\tilde{\mu}/2\}$.

Now we turn to the Fredholm determinant and determine a candidate for the pointwise limit of the kernel. The kernel $J^{\Gamma}_{\mu}(\eta,\eta')$ is given by an integral whose integrand has four main components: An exponential

$$\exp\{\Lambda(\zeta)-\Lambda(\eta')\},$$

a rational function (we include the differential with this term for scaling purposes)

$$d\zeta/\eta'(\zeta-\eta),$$

a doubly infinite sum

$$\mu f(\mu,\zeta/\eta'),$$

and an infinite product

$$g(\eta')/g(\zeta).$$

We proceed by the method of steepest descent, so in order to determine the region along the ζ and η contours which affects the asymptotics we must consider the exponential term first. The argument of the exponential is given by $\Lambda(\zeta)-\Lambda(\eta')$ where

$$\Lambda(\zeta) = -x\log(1-\zeta) + \frac{t\zeta}{1-\zeta} + m\log(\zeta).$$

For small ϵ, $\Lambda(\zeta)$ has a critical point in an $\epsilon^{1/2}$ neighborhood of -1. One has some freedom in where to center the expansion and for purposes of having a nice ultimate answer we choose to use

$$\xi = -1 - 2\epsilon^{1/2}\frac{X}{T}$$

We can rewrite the argument of the exponential as $(\Lambda(\zeta)-\Lambda(\xi)) - (\Lambda(\eta')-\Lambda(\xi)) = \Psi(\zeta)-\Psi(\eta')$. The idea of extracting asymptotics for this term (which starts like those done in [TW09a] but quickly becomes more involved due to the fact that τ tends to 1 as ϵ goes to zero) is then to deform the ζ and η contours to lie along curves such that outside the scale $\epsilon^{1/2}$ around ξ, $\mathrm{Re}\Psi(\zeta)$ is very negative, and $\mathrm{Re}\Psi(\eta')$ is very positive, and hence the contribution from those parts of the contours is negligible. Rescaling around ξ to blow up this $\epsilon^{1/2}$ scale, gives us the asymptotic exponential term. This change of variables sets the scale at which we should analyze the other three terms in the integrand for the J kernel.

Returning to $\Psi(\zeta)$, we make a Taylor expansion around ξ and find that in a neighborhood of ξ

$$\Psi(\zeta) \approx -\frac{T}{48}\epsilon^{-3/2}(\zeta-\xi)^3 + \frac{a}{2}\epsilon^{-1/2}(\zeta-\xi).$$

This suggests the following change of variables

$$\tilde{\zeta} = 2^{-4/3}\epsilon^{-1/2}(\zeta - \xi) \qquad \tilde{\eta} = 2^{-4/3}\epsilon^{-1/2}(\eta - \xi) \qquad \tilde{\eta}' = 2^{-4/3}\epsilon^{-1/2}(\eta' - \xi), \tag{263}$$

after which our Taylor expansion takes the form

$$\Psi(\tilde{\zeta}) \approx -\frac{T}{3}\tilde{\zeta}^3 + 2^{1/3}a\tilde{\zeta}.$$

In the spirit of steepest descent analysis we would like the ζ contour to leave ξ in a direction where this Taylor expansion is decreasing rapidly. This is accomplished by leaving at an angle $\pm 2\pi/3$. Likewise, since $\Psi(\eta)$ should increase rapidly, η should leave ξ at angle $\pm\pi/3$. Since $\rho_+ = 1/2$, $\alpha = 1$ which means that the ζ contour is originally on a circle of diameter $[-1+\delta, 1+\delta]$ and the η contour on a circle of diameter $[-1+2\delta, 1-\delta]$ for some positive δ (which can and should depend on ϵ so as to ensure that $|\zeta/\eta| \in (1, \tau^{-1})$). In order to deform these contours to their steepest descent contours without changing the value of the determinant, great care must be taken to avoid the poles of f, which occur whenever $\zeta/\eta' = \tau^k$, $k \in \mathbb{Z}$, and the poles of $1/g$, which occur whenever $\zeta = -\tau^{-n}$, $n \geq 0$. We will ignore these considerations in the formal calculation. The one very important consideration in this deformation, even formally, is that we must end up with contours which lie to the right of the poles of the $1/g$ function.

Let us now assume that we can deform our contours to curves along which Ψ rapidly decays in ζ and increases in η, as we move along them away from ξ. If we apply the change of variables in (263) the straight part of our contours become infinite rays at angles $\pm 2\pi/3$ and $\pm\pi/3$ which we call $\tilde{\Gamma}_\zeta$ and $\tilde{\Gamma}_\eta$[7].

Applying this change of variables to the kernel of the Fredholm determinant changes the L^2 space and hence we must multiply the kernel by the Jacobian term $2^{4/3}\epsilon^{1/2}$. We will include this term with the $\mu f(\mu, z)$ term and take the $\epsilon \to 0$ limit of that product.

Before we consider that term, however, it is worth looking at the new infinite product term $g(\eta')/g(\zeta')$. In order to do that let us consider the following. Set

$$q = 1 - r, \quad a = \frac{\log \alpha(c - xr)}{\log q}, \quad b = \frac{\log \alpha(c - yr)}{\log q}.$$

Then observe that

$$\begin{aligned}\prod_{n=0}^{\infty} \frac{1 + (1-r)^n \alpha(-c + xr)}{1 + (1-r)^n \alpha(-c + yr)} &= \frac{(q^a;q)_\infty}{(q^b;q)_\infty} = \frac{\Gamma_q(b)}{\Gamma_q(a)}(1-q)^{b-a} \\ &= \frac{\Gamma_q(b)}{\Gamma_q(a)} e^{(b-a)\log r} = \frac{\Gamma_{1-r}(-r^{-1}\log(\alpha c) + c^{-1}y + o(r))}{\Gamma_{1-r}(-r^{-1}\log(\alpha c) + c^{-1}x + o(r))} e^{c^{-1}(y-x)\log r + o(r \log r)},\end{aligned} \tag{264}$$

where the q-Gamma function and the q-Pochhammer symbols are given by

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}$$

when $|q| < 1$ and

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots.$$

[7]Note that this is *not* the actual definition of the contours which one uses in the real proof because of the singularity problem.

The notation $o(f(r))$ above refers to a function $f'(r)$ such that $f'(r)/f(r) \to 0$ as $r \to 0$. The q-Gamma function converges to the usual Gamma function as $q \to 1$, uniformly on compact sets (see [AAR99]).

Now consider the g terms and observe that in the rescaled variables this corresponds with (264) with $r = 2\epsilon^{1/2}$, $c = 1$ (recall $\alpha = 1$ as well) and

$$y = 2^{1/3}\tilde{\zeta} - \frac{X}{T}, \qquad x = 2^{1/3}\tilde{\eta}' - \frac{X}{T}$$

Since $\alpha c = 1$ and since we are away from the poles and zeros of the Gamma functions we find that

$$\frac{g(\eta')}{g(\zeta)} \to \frac{\Gamma\left(2^{1/3}\tilde{\zeta} - \frac{X}{T}\right)}{\Gamma\left(2^{1/3}\tilde{\eta}' - \frac{X}{T}\right)} \exp\left\{2^{1/3}(\tilde{\zeta} - \tilde{\eta}')\log(2\epsilon^{1/2})\right\}. \tag{265}$$

This exponential can be rewritten as

$$\exp\left\{\frac{\tilde{z}}{4}\log\epsilon\right\}\exp\left\{2^{1/3}\log(2)(\tilde{\zeta} - \tilde{\eta}')\right\}. \tag{266}$$

where

$$\tilde{z} = 2^{4/3}(\tilde{\zeta} - \tilde{\eta}'). \tag{267}$$

It appears that there is a problem in these asymptotics as ϵ goes to zero, however we will find that this apparent divergence exactly cancels with a similar term in the doubly infinite summation term asymptotics. We will now show how that $\log\epsilon$ in the exponent can be absorbed into the $2^{4/3}\epsilon^{1/2}\mu f(\mu, \zeta/\eta')$ term. Recall

$$\mu f(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\mu\tau^k}{1 - \tau^k\mu} z^k.$$

If we let $n_0 = \lfloor \log(\epsilon^{-1/2})/\log(\tau) \rfloor$ then observe that

$$\mu f(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\mu\tau^{k+n_0}}{1 - \tau^{k+n_0}\mu} z^{k+n_0} = z^{n_0}\tau^{n_0}\mu \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \tau^k\tau^{n_0}\mu} z^k.$$

By the choice of n_0, $\tau^{n_0} \approx \epsilon^{-1/2}$ so

$$\mu f(\mu, z) \approx z^{n_0}\tilde{\mu} f(\tilde{\mu}, z).$$

The discussion on the exponential term indicates that it suffices to understand the behavior of this function only in the region where ζ and η' are within a neighborhood of ξ of order $\epsilon^{1/2}$. Equivalently, letting $z = \zeta/\eta'$, it suffices to understand $\mu f(\mu, z) \approx z^{n_0}\tilde{\mu} f(\tilde{\mu}, z)$ for

$$z = \frac{\zeta}{\eta'} = \frac{\xi + 2^{4/3}\epsilon^{1/2}\tilde{\zeta}}{\xi + 2^{4/3}\epsilon^{1/2}\tilde{\eta}'} \approx 1 - \epsilon^{1/2}\tilde{z}.$$

Let us now consider z^{n_0} using the fact that $\log\tau \approx -2\epsilon^{1/2}$:

$$z^{n_0} \approx (1 - \epsilon^{1/2}\tilde{z})^{\epsilon^{-1/2}\frac{1}{4}\log\epsilon} \approx e^{-\frac{1}{4}\tilde{z}\log\epsilon}.$$

Plugging back in the value of $\tilde{z}$ in terms of $\tilde{\zeta}$ and $\tilde{\eta}'$ we see that this prefactor of z^{n_0} exactly cancels the $\log\epsilon$ term which came from the g infinite product term.

What remains is to determine the limit of $2^{4/3}\epsilon^{1/2}\tilde{\mu}f(\tilde{\mu},z)$ as ϵ goes to zero and for $z\approx 1-\epsilon^{1/2}\tilde{z}$. This limit can be found by interpreting the infinite sum as a Riemann sum approximation for an appropriate integral. Define $t=k\epsilon^{1/2}$, then observe that

$$\epsilon^{1/2}\tilde{\mu}f(\tilde{\mu},z)=\sum_{k=-\infty}^{\infty}\frac{\tilde{\mu}\tau^{t\epsilon^{-1/2}}z^{t\epsilon^{-1/2}}}{1-\tilde{\mu}\tau^{t\epsilon^{-1/2}}}\epsilon^{1/2}\to\int_{-\infty}^{\infty}\frac{\tilde{\mu}e^{-2t}e^{-\tilde{z}t}}{1-\tilde{\mu}e^{-2t}}dt.$$

This used the fact that $\tau^{t\epsilon^{-1/2}}\to e^{-2t}$ and that $z^{t\epsilon^{-1/2}}\to e^{-\tilde{z}t}$, which hold at least pointwise in t. If we change variables of t to $t/2$ and multiply the top and bottom by e^{-t} then we find that

$$2^{4/3}\epsilon^{1/2}\mu f(\mu,\zeta/\eta')\to 2^{1/3}\int_{-\infty}^{\infty}\frac{\tilde{\mu}e^{-\tilde{z}t/2}}{e^t-\tilde{\mu}}dt.$$

As far as the final term, the rational expression, under the change of variables and zooming in on ξ, the factor of $1/\eta'$ goes to -1 and the $\frac{d\zeta}{\zeta-\eta'}$ goes to $\frac{d\tilde{\zeta}}{\tilde{\zeta}-\tilde{\eta}'}$.

Therefore we formally find the following kernel: $-K_{a'}^{\mathrm{csc},\Gamma}(\tilde{\eta},\tilde{\eta}')$ acting on $L^2(\tilde{\Gamma}_\eta)$ where:

$$K_{a'}^{\mathrm{csc},\Gamma}(\tilde{\eta},\tilde{\eta}')=$$
$$\int_{\tilde{\Gamma}_\zeta}\exp\{-\frac{T}{3}(\tilde{\zeta}^3-\tilde{\eta}'^3)+2^{1/3}a'(\tilde{\zeta}-\tilde{\eta}')\}2^{1/3}\left(\int_{-\infty}^{\infty}\frac{\tilde{\mu}e^{-2^{1/3}t(\tilde{\zeta}-\tilde{\eta}')}}{e^t-\tilde{\mu}}dt\right)\frac{\Gamma\left(2^{1/3}\tilde{\zeta}-\frac{X}{T}\right)}{\Gamma\left(2^{1/3}\tilde{\eta}'-\frac{X}{T}\right)}\frac{d\tilde{\zeta}}{\tilde{\zeta}-\tilde{\eta}},$$

where $a'=a+\log 2$ (recall that this $\log 2$ came from (266)).

We have the identity

$$\int_{-\infty}^{\infty}\frac{\tilde{\mu}e^{-\tilde{z}t/2}}{e^t-\tilde{\mu}}dt=(-\tilde{\mu})^{-\tilde{z}/2}\pi\csc(\pi\tilde{z}/2),\tag{268}$$

where the branch cut in $\tilde{\mu}$ is along the positive real axis, hence $(-\tilde{\mu})^{-\tilde{z}/2}=e^{-\log(-\tilde{\mu})\tilde{z}/2}$ where log is taken with the standard branch cut along the negative real axis. We may use the identity to rewrite the kernel as

$$K_{a'}^{\mathrm{csc},\Gamma}(\tilde{\eta},\tilde{\eta}')=$$
$$\int_{\tilde{\Gamma}_\zeta}\exp\{-\frac{T}{3}(\tilde{\zeta}^3-\tilde{\eta}'^3)+2^{1/3}a'(\tilde{\zeta}-\tilde{\eta}')\}2^{1/3}\frac{\pi(-\tilde{\mu})^{-2^{1/3}(\tilde{\zeta}-\tilde{\eta}')}}{\sin(\pi 2^{1/3}(\tilde{\zeta}-\tilde{\eta}'))}\frac{\Gamma\left(2^{1/3}\tilde{\zeta}-\frac{X}{T}\right)}{\Gamma\left(2^{1/3}\tilde{\eta}'-\frac{X}{T}\right)}\frac{d\tilde{\zeta}}{\tilde{\zeta}-\tilde{\eta}}.$$

To make this cleaner we replace $\tilde{\mu}/2$ with $\tilde{\mu}$. Taking into account this change of variables (it also changes the $\exp\{-\tilde{\mu}/2\}$ in front of the determinant to $\exp\{-\tilde{\mu}\}$), we find that our final answer is

$$\int_{\tilde{\mathcal{C}}}e^{-\tilde{\mu}}\frac{d\tilde{\mu}}{\tilde{\mu}}\det(I-K_a^{\mathrm{csc},\Gamma})_{L^2(\tilde{\Gamma}_\eta)},$$

which is a simple transformation of (205). The details can be found in [CQ11a].

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Jeremy Quastel, Departments of Mathematics and Statistics, University of Toronto, 40 St George Street, Toronto, ON M5S 1L2

E-mail address: quastel@math.toronto.edu