$$\overset{?}{\Rightarrow} \quad \overline{\lim_{N\to\infty}} \frac{\log |u(t, \gamma N)|}{(\log N)^{2/3}} > 0 \quad a.s.$$

$$\Rightarrow \quad \overline{\lim_{|x|\to\infty}} \frac{\log |u(t,x)|}{(\log |x|)^{2/3}} > 0 \quad . \quad \blacksquare$$

24/10/16

LECTURE - 9.

RECALL: $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sigma(u)\xi . \quad u(0,x) = \psi(x).$

ψ - bdd m'ble, $\quad |\sigma(x) - \sigma(y)| \le \lambda |x-y|$

$x \in \mathbb{R}, \ t > 0.$

$$\Leftrightarrow \quad u(t,x) = P_t * \psi(x) + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\sigma(u(s,y))\xi(dy, ds) \quad - (*)$$

unique sol^n, continuous, $\mathcal{L}^p$, $\mathbb{E}|u(t,x)|^k \le L^k e^{k^3 t}$

INTERMITTENCY:

$\underline{\gamma}_k(x) = \underline{\lim}_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u(t,x)|^k$ - lower Lyapunov exponent

$\overline{\gamma}_k(x) = \overline{\lim}_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u(t,x)|^k$ - upper Lyapunov exp.

$\gamma_k(\cdot)$ are fns. of x, $\sup_{x\in\mathbb{R}} \overline{\gamma}_k(x) = O(k^3)$.

$\gamma_2(x) > 0$ then $\begin{cases} k \to \frac{1}{k}\underline{\gamma}_k(x) \\ k \to \frac{1}{k}\overline{\gamma}_k(x) \end{cases}$ are strictly increasing on $[2,\infty)$

$\Downarrow$? seperation of scales

Thm

$$0 < \overline{\lim_{N\to\infty}} \frac{1}{N} \max_{1\le l\le \exp(\theta_i N)} \log u(N,l) < \underline{\lim}_{N\to\infty} \frac{1}{N} \max_{1\le l\le \exp(\theta_{i+1} N)} \log u(N,l) < \infty$$

$\exists \ \theta_{i+1} > \theta_i, \quad i \ge 1.$

Theorem * $\inf_{z \in \mathbb{R}} |\psi(z)| > 0 \Rightarrow \inf_{z \in \mathbb{R}} \gamma_2(z) \geq \frac{L_\sigma^4}{4}$

where $L_\sigma = \inf_{z \in \mathbb{R}} \left| \frac{\sigma(z)}{z} \right|$. (Proof using (+) & renewal fn)

GP THEOREM $\psi : \mathbb{R} \to \mathbb{R}$, $\sigma(0) = 0$. , compact support & $\psi > 0$ on $(a,b) \subseteq (0,\infty)$.

$$I(\alpha) = \overline{\lim_{t \to \infty}} \frac{1}{t} \sup_{|x| > \alpha t} \log \mathbb{E}|u(t,x)|^2$$

- $I(\alpha) < 0$ if $\alpha > \lambda^2/2$
- $L_\sigma > 0$ then $\exists \alpha_0 > 0 \ni I(\alpha) > 0$ if $\alpha \in (0, \alpha_0)$.

THEOREM $\sigma(u) = u$ (P.A.M), $\psi(x) = 1$, $x \in \mathbb{R}$

Fix $t > 0$, $0 < \overline{\lim_{|x| \to \infty}} \frac{\log u(t,x)}{(\log(x))^{2/3}} < \infty$

$\mathbb{P}(u(t,x) > 0) = 1$ (Comparison principle).

COUPLING | $u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) u(s,y) \xi(dy, ds)$

Because p_{t-s} is the heat kernel & decays fast

" $\Rightarrow$ " $|x_1 - x_2| > 2\sqrt{t}$, $u(t,x_1)$ & $u(t,x_2)$ are "indep."

HEURISTIC.

THM | Fix $t > 0$, $k \geq 1$. $\exists \varphi > 0$ (depending on t, k) & $\{Y(x)\}_{x \in \mathbb{R}}$ with the following property.

$\{Y(t,x)\}$

(1) $\sup_{x \in \mathbb{R}} \mathbb{E}|u(t,x) - Y(x)|^k \leq \varphi e^{-\varphi k^3}$

(2) $\{x_i\}_{i=1}^m$ $|x_i - x_j| > 2k^3\sqrt{t}$ then $\forall i \neq j$, & then $\{Y(x_i)\}_{i=1}^m$ are independent

Proof: . $\alpha > 1$ $\quad U_t^{(\alpha)}(x) = 1 + \int_0^t \int_{x-\sqrt{\alpha t}}^{x+\sqrt{\alpha t}} p_{t-s}(x-y)\, U_s^{(\alpha)}(y)\, \xi(dy, ds)$

$$U_0^{(\alpha)}(x) = 1 .$$

Existence & uniqueness via Picard's iteration as^{m} weak sol^{n}

Step 0: $\quad U_t^{\alpha,n+1}(x) = 1 + \int_0^t \int_{x-\sqrt{\alpha t}}^{x+\sqrt{\alpha t}} p_{t-s}(x-y)\, U_s^{\alpha,n}(y)\, \xi(dy, ds)$, $n \geq 1$. $\quad - (*)$.

Set $\quad u_t^{\alpha,0}(x) \equiv 1$.

$n=1$, $\quad u_t^{\alpha,1}(x) = 1 + \int_0^t \int_{x-\sqrt{\alpha t}}^{x+\sqrt{\alpha t}} p_{t-s}(x-y)\, \xi(dy, ds)$.

$|x_1 - x_2| > 2\sqrt{\alpha t}$ $\quad u_t^{\alpha,1}(x_1)$ & $u_t^{\alpha,1}(x_2)$ are indep.

Induction $\quad |x_i - x_j| > 2\sqrt{\alpha t}$, $\{u_t^{\alpha,1}(x_i)\}_{i=1}^{m}$ are independent.

$\Big\downarrow$ m'ble - Dalang's result

$|x_i - x_j| > 2n\sqrt{\alpha t}$, $\{u_t^{\alpha,n}(x_i)\}_{i=1}^{m}$ are independent.

Step 1: $\quad \sup_{\alpha > 0} \sup_{x \in \mathbb{R}} \mathbb{E} |u_t^{\alpha,n}(x) - u_t^{\alpha,n+1}(x)|^k \leq \dfrac{e^{128k^3 t}}{2^{k(n-1)}}$

$\Rightarrow \quad u_t^{\alpha}(x) = \lim_{n \to \infty} u_t^{\alpha,n}(x)$ exists in L^{β} $(\beta = 128k^3)$

& solves $(*)$

Step 2 $\quad \sup_{x \in \mathbb{R}} \mathbb{E}\, |u(t,x) - u_t^{\alpha}(x)|^k \leq M^k e^{Mk^3 t - \frac{\alpha k}{2}}$ (Proof needed)

FINAL $\quad \{Y(x)\}_{x \in \mathbb{R}}$ $\quad Y(x) = u_t^{\alpha,n}$ $\quad$ (α, n to be chosen later)

$$\mathbb{E}\,|u(t,x) - Y(x)|^k \leq 2^k \Big[\mathbb{E}\,|u(t,x) - u_t^{\alpha}(x)|^k + \mathbb{E}\,|u_t^{\alpha}(x) - u_t^{\alpha,n}(x)|^k \Big]$$

$$\leq 2^k \Big[M^k e^{Mk^3 t - \frac{\alpha k}{2}} + \|u_t^{\alpha}(x) - U_t^{\alpha,n}(x)\|_k^k \Big]$$

$$\le 2^k \left[M^k e^{Mk^3 t - \frac{\alpha k}{2}} + \left(\sum_{l=n}^{\infty} \| u_t^{\alpha, l+1}(x) - u_t^{\alpha, l}(x) \|_k \right)^k \right]$$

$$\le 2^k \left[M^k e^{Mk^3 t - \frac{\alpha k}{2}} + \left(\sum_{l=n}^{\infty} \frac{e^{128 k^2 t}}{2^{l+1}} \right)^k \right]$$

$$\le 2^k \left[M^k e^{Mk^3 t - \frac{\alpha k}{2}} + e^{128 k^3 t} \cdot e^{-\frac{nk}{2}} \right]$$

$\alpha = c_1 k^2$, $n = c_2 k^2$ c_1, c_2 - can depend on t.

---

not quite correct.

$$\le \varphi e^{-\varphi^3 t} \qquad \varphi > 0, \text{ constant depending on } t.$$

Missing details:

$\Rightarrow$ $Y(x) = \{ u_t^{c_1 k^2, c_2 k^2}(x) \} \Rightarrow |x_i - x_j| > 2k^3 \sqrt{t}$ $\boxed{(???)}$

FACT / CONSEQUENCE

$$\mathcal{L}(m) = \left\{ \{Y(x)\}_{x \in \mathbb{R}} \;\middle|\; \begin{array}{l} Y(x) - \text{st. random field} \\ \forall l, \{Y(x_i)\}_{i=1}^{l} \; |x_i - x_j| > m \Rightarrow \text{indep.} \end{array} \right\}$$

CORRELATION LENGTH : $t > 0$

$$L_X(\eta, \delta) = \inf \left\{ m \ge 0 \;\middle|\; \inf_{Y \in \mathcal{L}(m)} \sup_{x \in \mathbb{R}} \mathbb{P}(|X(t,x) - Y(x)| > \delta) < \eta \right\}$$

$$\sup_{x \in \mathbb{R}} \mathbb{P}(|u(t,x) - Y(x)| > \delta) \le \delta^{-k} \varphi e^{-\varphi k^3}, \quad m = 2k^3\sqrt{t}$$

choose $\delta = e^{-c_1 k^2} \Rightarrow \eta = e^{-c_2 k^3}$

$$L_u(e^{-c_2 k^3}, e^{-c_1 k^2}) \le c_3 k^3 = 2\sqrt{t}\, k^3.$$

THM: $t \ge 0$ fixed. As $\varepsilon \to 0$, $L_{u_t}(\varepsilon, e^{-c_1 (|\log \varepsilon|)^{2/3}}) = O(|\log \varepsilon|)$

INTERMITTENCY ISLANDS

$b > a > 0$, $t > 0$, $[c,d]$ interval is ~~an~~ (a,b) - Intermittency island.

(1) $u(t,c) = u(t,d) = a$

(2) $\sup_{x \in [c,d]} u(t,x) > b$ (3) $u(t,x) > a$ for $x \in (c,d)$

LEMMA: $\sigma(u) = u$, $\psi(x) = 1$. Given ~~$b > a > 0$~~, $\varepsilon \in (0,1)$, $t > 0$ $\exists$ $b > a > 0$

$$\mathbb{P}(\exists \ (a,b)\text{- intermittency island at time } t) > 1 - \varepsilon.$$

Proof:

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\, u(s,y)\, \xi(dy,ds)$$

$$\mathbb{E}[u(t,x)] = 1 \quad \& \quad \mathbb{P}(u(t,x) > 0) = 1.$$

$$\varliminf_{|x| \to \infty} u(t,x) < \infty \quad a.s.$$

$\exists$ $a > 0$, $\mathbb{P}\left(\varliminf_{|x| \to \infty} u(t,x) < \frac{a}{2}\right) > 1 - \varepsilon.$

We also know $\varlimsup_{|x| \to \infty} u(t,x) = \infty$ a.s.

$\Rightarrow$ ~~$\forall$~~ $b > a$. $\mathbb{P}\left(\varlimsup_{|x| \to \infty} u(t,x) > b\right) = 1$.

continuity + IVT $\Rightarrow$ ■ [stronger implication $\to$ b ind. of ε; ∞'ly many (a,b) islands]

$J_t(a,b;R) :=$ length of largest (a,b) island in $[0,R]$

THM $1 < a < b$, Assume $\mathbb{P}(u(t,0) > b) > 0$.

$$\varlimsup_{R \to \infty} \frac{J_t(a,b;R)}{[\log R]^2} < \infty$$

Proof: $u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\, u(s,y)\, \xi(dy,ds)$

$\mathbb{E}[u(t,0)] = 1$

Suppose $u(t,0) = 1$ a.s. $\Rightarrow$

$$1 = 1 + \int_0^t \int_{\mathbb{R}} p_{t-s}(-y)\, u(s,y)\, \xi(dy,ds)$$

$$\Rightarrow \int_0^t \int_{\mathbb{R}} p_{t-s}(-y)\, u(s,y)\, \xi(dy,ds) = 0$$

$\Rightarrow u(s,y) = 0$ a.s., $s \in (0,t)$, $y \in \mathbb{R}$.

$s \to 0$

$\Rightarrow u(0,y) = 0$ a.s. $\Rightarrow \Leftarrow$ $\psi = 1$.

Thus $P(u(t,0) > b) > 0$ for some $b > 1$.

$\varepsilon = CR^{-m}$ choose c, R, m.

Let $0 < \delta < 1$ be small enough

- $a - 2\delta > 1$ $\quad \mathbb{P}(u(t,0) \geq b + 2\delta) > 0$

$\mathbb{P}(|u(t,x) - Y(x)| > \delta) < \frac{c}{R^m}$ $\forall x \in \mathbb{R}$.

$Y(\cdot) \in \mathcal{L}(c_1 |\log \varepsilon|)$ (previous result).

$x_j = c_2 j \log R \Rightarrow \{Y(x_j)\}_{j \geq 1}$ are indep.

$$\mathbb{P}\Big(\max_{0 \leq j \leq \lfloor \frac{R}{\log R} \rfloor} |u(t,x_j) - Y(x_j)| > \delta\Big) \leq \frac{c}{R^{m-1}}$$

j is good. $Y(x_j) < a - \delta$, $Y(x_{j+1}) > b + \delta$, $Y(x_{j+2}) < a - \delta$.

$p = P(j \text{ good})$ $\quad \mathbb{P}(Y_{x_j} < a - \delta, Y_{x_{j+1}} > b + \delta, Y_{x_{j+2}} > a - \delta)$

$$= \left(\mathbb{P}(Y_0 < a-\delta)\right)^2 \; \mathbb{P}(Y_0 > b+\delta)$$

$$\geq \left(\mathbb{P}(u(t,0) < a-2\delta) - \frac{c}{R^m}\right)^2 \left(\mathbb{P}(u(t,0) > b+2\delta) - \frac{c}{R^m}\right)$$

Take m-large, $R > 0 \Rightarrow p > 0$.

$$\mathbb{P}(j, j+3, \quad , j+3n \text{ are all bad}) = (1-p)^n$$

$$P\left(\exists j \quad 0 < j \leq \left\lfloor \frac{R}{\log R} \right\rfloor, \; j, j+3, \dots, j+3\lfloor \gamma \log R \rfloor \text{ are all bad}\right)$$

$$\lesssim \frac{R}{\log R}(1-p)^{\lfloor \gamma \log R \rfloor} \leq \frac{c}{R^2} \quad \text{for } \gamma\text{-large enough.}$$

$$\Rightarrow \sum_{R=1}^{\infty} P(\cdot \;) < \infty$$

(using B. C.)

$$\mathbb{P}\left(J_t(a,b;R) < c_2(\log R)^2 \text{ for } R \text{ sufficiently large}\right) = 1.$$