$$P\left(\max_{1\le l\le \exp(\theta N)} \psi_N(l) \ge \delta \|\psi_N(0)\|_M\right) \le \sum_{l=1}$$

CHEBYSHEV + UNION

$$P\left(\max_{1\le l\le \exp(\theta N)} \psi_N(l) \ge \delta \|\psi_N(0)\|_M\right) \le \sum_{l=1}^{\exp(\theta N)} P(\psi_N(l) \ge \delta \|\psi_N(0)\|_m)$$

$$\le (1+e^{\theta n}) \frac{\mathbb{E}\,\psi_N(0)}{\delta \|\psi_N(0)\|_m}$$

$$= e^{-N\left(\frac{\gamma(m)}{m} - \theta\right) + o(N)}$$

if $\theta < \frac{\gamma(m)}{m} \Rightarrow \overline{\lim_{N\to\infty}} \frac{1}{N} \max_{1\le l\le \exp(\theta N)} \log \psi_N(l) < \frac{\gamma(m)}{m}$

$\exists\ \theta_j \quad \theta_j < \theta_{j+1}$

$$0 < \overline{\lim_{N\to\infty}} \frac{1}{N} \max_{1\le l\le \exp(\theta_j N)} \log \psi_n(l) < \lim_{N\to\infty} \max_{1\le l\le \exp(\theta_{j+1} N)} \log \psi_N(l) < \infty$$

if $\frac{\gamma(m)}{m}$ is strictly increasing.

N large "$x \to \psi_N(x)$"
increasing large peaks on exp. scale.

LECTURE - 8

10/10/16

Recall: $u(t,x) = P_t * \psi(x) + \lambda \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\, u(s,y)\, \xi(dy,ds)$

CORRECTION.

$$\mathbb{E} \prod_{j=1}^{k} u(t,x_j) = \mathbb{E} \prod_{j=1}^{k} \psi(X_t^j)\, e^{-L_t^X} \quad \text{—} (*)$$

$L_t^X = \sum_{1\le i\ne j\le k} L_t^{(i,j)}$ local time $\{X_t^j\}$ indep. BM starting at x_j.

- $\mathbb{E}|u(t,x)|^k \geq \mathbb{E}\prod_{j=1}^{k} u(t,x_j) \qquad x_j \equiv x \ \forall j$

$$\geq C^k \exp\left[\frac{\lambda^4 (k(k^2-1))t}{24}\right]$$

when $\psi(y) \geq C > 0$.

$$\frac{du}{dt} = \Delta u + \lambda\sigma(u)\xi \qquad u(0,x) = \psi(x).$$

$\gamma_k(x) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}|u(t,x)|^k$ — lower Lyapunov exp.

σ - glob. Lipschitz. ψ - bdd m'ble.

$\bar{\gamma}_k(x) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \mathbb{E}|u(t,x)|^k$ — upper Lyapunov exp.

$\mathbb{E}|u(t,x)|^k \leq L^k e^{Lk^3 t} \quad \rightarrow \quad \gamma_k = O(k^3)$

$u \in \mathcal{L}^{\beta,2}$, $\forall$ $r \geq 1$, $t > 0$, $x \in \mathbb{R}$

- Similar proposition like in non-negative stationary field. setting

If $\gamma_2(x) > 0$ then $\frac{1}{k}\gamma_k(x)$ is strictly increasing in k.

for $k \in [2,\infty)$ $\frac{1}{k}\bar{\gamma}_k(x)$ is strictly increasing in k.

DEFINITION $\inf_{x\in\mathbb{R}} \psi(x) > 0$ & $\gamma_2(x) > 0$ $\forall x \in \mathbb{R}$ then

we say u is weakly intermittent.

THEOREM: Assume $\inf_{z\in\mathbb{R}} \psi(z) > 0$. Then

$$\inf_{x\in\mathbb{R}} \gamma_2(x) \geq \frac{(L_{\sigma})^4}{4} \quad \text{where}$$

$L_\sigma = \inf_{z \in \mathbb{R}} \left| \frac{\sigma(z)}{z} \right|$. Therefore u is weakly intermittent if $\inf_{z \in \mathbb{R}} \psi(z) > 0$ & $L_\sigma > 0$.

Proof: $u(t,x) = P_t * \psi(x) + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\, \sigma(u(s,y))\, \xi(dy, ds)$

$$\Rightarrow \quad \mathbb{E}[u(t,x)^2] = [P_t * \psi(x)]^2 + \int_0^t \int_{\mathbb{R}} P_{t-s}^2(x-y)\, \mathbb{E}[\sigma^2(u(s,y))]\, dy\, ds$$

$$\geq \inf_{z \in \mathbb{R}} \psi^2(z) + L_\sigma^2 \int_0^t \int_{\mathbb{R}} P_{t-s}^2(x-y)\, \mathbb{E}[u(s,y)^2]\, dy\, ds$$

Let $I(t) = \inf_{x \in \mathbb{R}} \mathbb{E}[u(t,x)^2]$, $t \geq 0$

$$\Rightarrow \quad I(t) \geq \inf_{z \in \mathbb{R}} \psi^2(z) + L_\sigma^2 \int_0^t \left[\int_{\mathbb{R}} P_{t-s}^2(x-y)\, dy \right] I(s)\, ds$$

$$\geq \inf_{z \in \mathbb{R}} \psi^2(z) + \frac{L_\sigma^2}{2\sqrt{\pi}} \int_0^t \frac{I(s)\, ds}{\sqrt{t-s}}$$

$I(\cdot)$ is a super solution of

$$f(t) = a + b \int_0^t ds\, f(s)\, g(t-s) \quad - (+)$$

$a = \inf_{z \in \mathbb{R}} \psi^2(z) > 0$, $\quad b = \frac{L_\sigma^2}{2\sqrt{\pi}}$ & $g(t) = \frac{1}{\sqrt{t}}$

PROPN: Under the assumption $\overline{\lim_{t \to \infty}} \frac{1}{t} \log f(t) < \infty$

(a) (+) has a unique sol^n & $\lim_{t \to \infty} e^{-\pi b^2 t} f(t) = a/b$

(b) F is a super solution to (+) & $\sup_{t \geq 0} e^{-\beta t} F(t) < \infty$ on some $\beta > 0$ $\Big\} \Rightarrow F(t) \geq f(t)$, $\forall t \geq 0$.

No proof of proposition.

$$u \in L^{\beta,2} \Rightarrow \sup_{t \geq 0} e^{-\beta t} I(t) < \infty$$

$$\Rightarrow I(t) \geq f(t) \quad \Rightarrow e^{-\pi b^2 t} I(t) \geq e^{-\pi b^2 t} f(t)$$

$$\Rightarrow \lim_{t \to \infty} e^{-\pi b^2 t} I(t) \geq a/b \quad - (++)$$

$$\gamma_2(x) = \lim_{t \to \infty} \log E|u(t,x)|^2 \geq \lim_{t \to \infty} \frac{1}{t} \log I(t)$$

$$\geq \pi b^2 = \frac{L_\sigma^4}{4}$$

∎

COR: $\inf_{x \in \mathbb{R}} \psi(x) > 0$ PAM. is w-Intermittent.

Intermittency Front: $\psi : \mathbb{R} \to \mathbb{R}$ bdd, compact support

$\exists\ (a,b) \subseteq \mathbb{R}$, $\inf_{x \in (a,b)} \psi(x) > 0$.

$\sigma : \mathbb{R} \to \mathbb{R}$, globally Lipschitz (λ), $\sigma(0) = 0$.

$$I(\alpha) = \overline{\lim_{t \to \infty}} \frac{1}{t} \sup_{|x| > \alpha t} \log E|u(t,x)|^2, \quad \alpha \geq 0.$$

THEOREM: Let u be a sol^n to

$$u(t,x) = P_t * \psi(x) + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\, \sigma(u(s,y))\, \xi(dy,ds)$$

with σ & ψ as above.

- $I(\alpha) < 0$ if $\alpha > \lambda^2/2$
- Suppose $L_\sigma > 0$, $\exists\ \alpha_0 > 0 \ni I(\alpha) > 0$ if $\alpha \in (0, \alpha_0)$.

Proof: Revisit later, may be ∎

REMARKS $\alpha < \alpha_0 \Rightarrow \sup_{|x|>\alpha t} E|u(t,x)|^2 \approx e^{c_1 t}$ as $t\to\infty$, $c_1>0$

$\alpha \geq \frac{\lambda^2}{t} \Rightarrow \sup_{|x|>\alpha t} \mathbb{E}|u(t,x)|^2 \sim e^{-c_2 t}$, $c_2>0$ as $t\to\infty$.

Does there exist α^* $\ni$ $I(\alpha) > 0$ $\alpha = \alpha^*$

$I(\alpha) < 0$ $\alpha > \alpha^*$

Conjecture $\alpha^* = \frac{\lambda^2}{2}$? [Chen - Dalang]

INTERMITTENCY ISLANDS:

$\psi(x) = 1$, $\sigma(u) = u$.

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\, u(s,y)\, \xi(dy,ds)$$

THEOREM $t>0$

$$0 < \overline{\lim_{|x|\to\infty}} \frac{\log|u(t,x)|}{(\log|x|)^{2/3}} < \infty$$

REMARK: • $|x|\to\infty$ "tallest" peak is of $e^{\Theta(\log|x|^{2/3})}$

Known $u(t,x) > 0$ a.s. $\forall t, x$.

• $h(t,x) = \log|u(t,x)| \Rightarrow$ "$\frac{\partial h}{\partial t} = c_1 \Delta h + (\nabla h)^2 + \xi$"

$\boxed{\cancel{\text{KPZ EQN}}}$

Cole - Hopf Transformation.

$$0 < \overline{\lim_{|x|\to\infty}} \frac{h(t,x)}{\log|x|^{2/3}} < \infty.$$

• (No limit) $\mathbb{E}[u(t,x)] = 1 \xrightarrow{\text{Fatou's lemma}} \mathbb{E}\left[\underline{\lim}_{|x|\to\infty} u(t,x)\right] \leq 1$

$\Rightarrow \quad \lim_{|x|\to\infty} u(t,x) < \infty$ a.s.

$\Rightarrow \quad \lim_{|x|\to\infty} \dfrac{\log|u(t,x)|}{(\log|x|)^{2/3}} \leq 0. \Rightarrow$ (S) $\overline{\lim} > 0 \geq \underline{\lim}$

& so no limit exists.

Proof of theorem:

Propn: $\forall\ t>0,\ -\infty < \underline{\lim}_{z\to\infty} \dfrac{\log P(u(t,x)>z)}{[\log(z)]^{3/2}}$

$\leq \overline{\lim}_{z\to\infty} \dfrac{\log P(u(t,x)>z)}{(\log z)^{3/2}} \leq 0.$

unif. over all $x \in \mathbb{R}$.

[Proof of Propn: later (S): $X = u(t,x)$ $E[X^k] = \Theta(L^k e^{Lk^3 t})$

$P(X \geq z) = ?$ Paley-Zygmund & Chebyshev. - Later]

$P\left(\max_{0\leq j\leq N} u(t,j) \geq e^{\alpha(\log N)^{2/3}}\right)$ $\alpha>0,\ N\geq 1$

(Propn. + Union bd.) $\leq (N+1) \sup_{y\in\mathbb{R}} P\left(u(t,y) > e^{\alpha(\log N)^{2/3}}\right)$

$= (N+1)\, K\, e^{-K\alpha \log N} < \dfrac{2K}{N^{1-K\alpha}}$

KOLMOGOROV- CONTINUITY ~~RES~~ THM: (long Proof)

$\sum_{n\geq 1} P\left(\max_{0\leq j\leq N} \sup_{x\in(j,j+1)} |u(t,x)-u(t,j)| \geq e^{\alpha(\log N)^{2/3}}\right) < \infty.$

for α large.

$$\Rightarrow \sum_{n=1}^{\infty} P\left(\sup_{x \in [0,N]} u(t,x) > e^{\alpha (\log N)^{2/3}} \right) < \infty \quad \text{for } \alpha \text{ large.}$$

Symmetry for $x \in [-N, 0]$ & Borel-Cantelli

$$\overline{\lim_{N \to \infty}} \sup_{|x| \le N} \frac{\log (u(t,x))}{(\log N)^{2/3}} < \infty \quad \text{a.s.}$$

$$\Rightarrow \overline{\lim_{|x| \to \infty}} \frac{\log u(t,x)}{(\log |x|)^{2/3}} < \infty \quad \text{a.s.}$$

$$\mathcal{E}(N, \alpha, \gamma) = \{ u(t, \gamma N) > e^{(\alpha \log N)^{2/3}} \} \quad \text{gap } \gamma > 0$$

$$\mathbb{P}(\mathcal{E}(N, \alpha, \gamma)) \ge c_1 e^{-\frac{1}{c_1} \alpha \log N} \quad c_1 \in (0,1).$$

$$\ge c_1 N^{-\alpha / c_1}$$

For $\alpha / c_1 < 1$, $\sum_{N=1}^{\infty} \mathbb{P}(\mathcal{E}(N, \alpha, \gamma)) = \infty$.

Generalized BC Propn:

PROPN: $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N$ be events (i) $\sum_{n=1}^{\infty} P(\mathcal{E}_n) = \infty$

(ii) $\exists\ \theta \in [1, \infty)$ $\ni$ $\forall\ j > i \ge 1$

$$\mathbb{P}(\mathcal{E}_j \mid \mathcal{E}_i) \le \theta\, \mathbb{P}(\mathcal{E}_j)$$

$$\Rightarrow \mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{N=n}^{\infty} \mathcal{E}_N \right) \ge \frac{1}{\theta}$$

Show $\forall\ \varepsilon > 0$, $\exists\ \gamma = \gamma(\varepsilon) > 0$

$$P(\mathcal{E}(j, \alpha, \gamma) \mid \mathcal{E}(i, \alpha, \gamma)) \le (1+\varepsilon)\, \mathbb{P}(\mathcal{E}(i, \alpha, \gamma)) \quad \forall\ j > i \ge 1.$$

$\Rightarrow$ By Generalized BC Propn

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{N=n}^{\infty} \mathcal{E}(N, \alpha, \gamma_\varepsilon) \right) \ge \frac{1}{1+\varepsilon}$$

$$\overset{?}{\Longrightarrow} \quad \overline{\lim_{N\to\infty}} \frac{\log |u(t,x_N)|}{(\log N)^{2/3}} > 0 \quad a.s.$$

$$\Longrightarrow \quad \overline{\lim_{|x|\to\infty}} \frac{\log |u(t,x)|}{(\log |x|)^{2/3}} > 0. \qquad \blacksquare$$