$\varepsilon\mathbb{Z} \ni \varepsilon i$
$i \in \mathbb{Z}$

$$dX_t^{\varepsilon,i} = \Delta^{(\varepsilon,D)} X_t^{\varepsilon,i}\, dt + \frac{1}{\sqrt{\varepsilon}}\, dB_t^{\varepsilon,i}$$

$$\Delta^{\varepsilon,D} X_t^{\varepsilon,i} = \frac{1}{\varepsilon^2}\left[X_t^{\varepsilon,i+1} + X_t^{\varepsilon,i-1} - 2X_t^{\varepsilon,i} \right]$$

"THEOREM".

$$\sup_{x\in[-N,N]} \sup_{t\in[0,T]} \left| X_t^{\varepsilon,\varepsilon[\frac{x}{\varepsilon}]} - u(t,x) \right| \longrightarrow 0 \text{ in Probability}$$

as $\varepsilon \to 0$.

[Initial conditions not discussed].

$i \in \mathbb{Z}$, $\quad Y_i^{t,\varepsilon} = u(t, \varepsilon i) = \int_0^t \int_{\mathbb{R}} p_{t-s}(\varepsilon i - y)\, \xi(dy\, ds)$

05/10/16

LECTURE - 7

ξ - Space-time white noise in $\mathbb{R}\times\mathbb{R}$.

$\{\Phi(t,x)\}_{t\geq 0, x\in\mathbb{R}}$ $\quad \|\Phi\|_{\beta,2} = \sup_{t\geq 0} \sup_{x\in\mathbb{R}} e^{-\beta t} \left[\mathbb{E}[\Phi(t,x)]^2\right]^{1/2}$

$$\mathcal{L}^{\beta,2} = \text{Span}\{\Phi \mid \Phi(t,x) = \mathbb{1}_{(a,b]}(t)\, \varphi(x)\, X \quad \varphi \text{ bdd m'ble} \quad X - \mathcal{F}_a \text{ m'ble}\}$$

$$\Rightarrow u(t,x) = p_t * \psi(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\, \sigma(u(s,y))\, \xi(dy\, ds) \quad \text{— } (*)$$

$$(\equiv) \quad \frac{du}{dt} = \Delta u + \sigma(u)\xi, \quad u(0,x) = \psi(x)$$

THM σ - globally Lipschitz $\Rightarrow u \in \mathcal{L}^{\beta,2}$ for some $\beta > 0$ satisfying $(*)$ ψ - bdd m'ble. u is a.s. unique among all random fields such that

$(+)$ - $\sup_{x\in\mathbb{R}} \mathbb{E}|u(t,x)|^k \leq L^k e^{Lk^3 t}$ $\quad t>0 \quad \forall k\geq 1$.

REMARK: • In proof of Theorem a sol^n of $(*)$ in $\mathcal{L}^{\beta,2}$ constructed & Choise of $\beta = c(\sigma,\psi)$

• This particular u satisfies $(+)$ with large enough $L = L(\sigma,\psi)$.

Uniqueness. $u, v \in \mathcal{L}^{\beta,2}$ satisfy ⊛ ~~since~~ then $u(t,x) = v(t,x)$.

INTERMITTENCY - (Next Goal)

u is said to be "intermittent" if u has large peaks over small islands for any given t.

PARABOLIC ANDERSON MODEL:

$$\frac{\partial u}{\partial t} = \Delta u + \lambda u \xi \qquad \lambda > 0$$

$$u(0,x) = \psi(x)$$

Want to show "k^3" in ⊕ is of correct order.

$\boxed{\text{THEOREM:}}$ Let u satisfy ⊛ with $\sigma(u) = \lambda u$, $\lambda \neq 0$

$$\inf_{y \in \mathbb{R}} \psi(y) > 0$$

$$\mathbb{E}|u(t,x)|^k \geq C^k e^{\frac{\lambda^4 k(k^2-1)t}{24}}, \quad \forall\, k \geq 2,\ t > 0,\ x \in \mathbb{R}.$$

Proof: uses moment duality. [Basic Ref: Ethier Kurtz]

$\{X_t\}_{t \geq 0}$; $X_0 = x$ $\quad \{Y_t\}_{t=0}$ $Y_0 = y$.

X & Y are dual with $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad \mathbb{E}[f(x, Y_t)] = \mathbb{E}[f(X_t, y)]$ $\forall\, t \geq 0$

Eg. $f(x,y) = y^x \Rightarrow \mathbb{E}[Y_t^{\,x}] = \mathbb{E}[y^{X_t}]$

$\boxed{\text{PROPOSITION:}}$ $k \geq 1$ & $t > 0$. $x_1, \dots, x_k \in \mathbb{R}$. u - solution to ⊛.

$\{X_t^i\}_{t \geq 0}$ indep BM starting at x_i.

$$\mathbb{E}\left[\prod_{j=1}^{k} |u(t,x_j)|\right] = \mathbb{E}\left[\prod_{i=1}^{k} \psi(X_t^j)\, e^{\lambda^2 \mathbb{L}_t^X}\right]$$

$$\mathbb{L}_t^X = \sum_{1 \leq i \neq j \leq k} L_t^{(i,j)} \qquad L_t^{i,j} = \text{local time at } 0 \text{ of } X^i - X^j.$$

Assume Proof of proposition. $\Rightarrow$ Proof of Theorem.

$$\mathbb{E}|u(t,x)|^k = \mathbb{E}\left[\prod_{j=1}^{k} \psi(X_t^j)\, e^{\lambda^2 \mathbb{L}_t^X}\right]$$

X_t^j - indep BM starting at x.

Let $c > 0$, $\psi(y) \geq c \quad \forall y \in \mathbb{R} \Rightarrow \mathbb{E}|u(t,x)|^k \geq c^k \, \mathbb{E} e^{\lambda^2 \mathbb{L}_t^x}$

$X_t^i - X_t^j \equiv$ BM starting at 0 with variance $2t \equiv W_t^{i,j}$

$$L_t^{(i,j)} = \frac{1}{2}|W_t^{i,j}| - \frac{1}{2}\int_0^t \operatorname{sgn}(W_s^{i,j})\, dW_s^{i,j}$$

$$\geq -\frac{1}{2}\int_0^t \operatorname{sgn}(W_s^{i,j})\, dW_s^{i,j}$$

$$\mathbb{L}_t^x \geq -\frac{1}{2}\sum_{1 \leq i < j \leq k} \int = -\frac{1}{4} \underbrace{\sum_{i=1}^k \sum_{j=1}^k \int_0^t \operatorname{sgn}(W_s^{i,j})\, dW_s^{i,j}}_{M_t}$$

$B_t^i = X_t^i - x, \quad 1 \leq i \leq k$

$$M_t \overset{?}{=} 2\sum_{j=1}^k \int_0^t \sum_{i=1}^k \operatorname{sgn}(B_s^i - B_s^j)\, dB_s^i$$ [To be shown later]

$$\sum_{i=1}^k \operatorname{sgn}(X_s^i - X_s^j)\, d(X_s^i - X_s^j)$$

$$\langle M \rangle_t = 4\sum_{j=1}^k \int_0^t \Big(\sum_{i=1}^k \operatorname{sgn}(B_s^i - B_s^j)\Big)^2 ds$$

$$= 4\sum_{j=1}^k \int_0^t \Big(\sum_{\substack{1 \leq i \leq k \\ i \neq j}} 2\,\mathbf{1}_{B_s^j > B_s^i} - 1\Big)^2 ds$$

Fact: $y_1, \dots, y_k$ (*) distinct. $\quad \sum_{j=1}^k \Big[\sum_{\substack{1 \leq i \leq k \\ i \neq j}} 2\,\mathbf{1}_{B_s^j > B_s^i} - 1\Big]^2 = \frac{k(k^2-1)}{3}$

$\Rightarrow \langle M \rangle_t = \frac{4k(k^2-1)t}{3} \qquad M_t \overset{d}{=} \sqrt{\frac{4k(k^2-1)}{3}}\, \tilde{B}_t$ BM

$$\Rightarrow \mathbb{E}\big[e^{\lambda^2 \mathbb{L}_t^x}\big] \geq \mathbb{E}\big[e^{-\frac{1}{4}\lambda^2 M_t}\big]$$

$$= e^{\frac{\lambda^4}{16} \frac{4k(k^2-1)}{3} \frac{t}{2}} = e^{\frac{\lambda^4 k(k^2-1)t}{24}}.$$

$\boxed{\text{INTERMITTENCY:}}$ $\{\psi_t(x)\}$ - non-negative random field that is stationary in x.

$$\gamma(k) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} |\psi_t(x)|^k$$

γ - Lyapunov exponent of ψ.

$$\mathbb{E}\left[|\psi_t(x)|^k \right]^{1/k} \le \mathbb{E}\left[|\psi_t(x)|^{k+1} \right]^{1/k+1} \quad (\text{Jensen's})$$

$$\Rightarrow \quad \frac{\gamma(k)}{k} \le \frac{\gamma(k+1)}{k+1} \quad \forall\ k \ge 1.$$

$\boxed{\text{DEFN}}$

$\psi_t(\cdot)$ is $\boxed{\text{intermittent}}$ if $k \longmapsto \frac{\gamma(k)}{k}$ is strictly increasing in $[2, \infty)$

[Convention : $\mathbb{E}|\psi_t(x)| = 1 \Rightarrow \gamma_1 = 0$].

$\boxed{\text{PROPOSITION:}}$ Assume $\gamma(k) < \infty$ for large enough k

→ γ - convex fn in $(0, \infty)$

→ If $\gamma(k_0) > 0$ for some $k_0 > 1$ then ψ is "intermittent" in $[k_0, \infty)$

Proof : (i) m, n $\alpha \in (0,1)$

$$\gamma(\alpha m + (1-\alpha)n) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} |\psi_t(x)|^{\alpha m + (1-\alpha) n}$$

$$\underset{\substack{\text{Holder on}\\ \psi}}{\le} \alpha\gamma(m) + (1-\alpha)\gamma(n).$$

[why $\gamma(k) < \infty$? needed]

(ii) $k_0 \le k < k'$ $\quad \alpha = \frac{k-1}{k'-1}$

$m = k'$, $n = 1$ $\quad \alpha m + (1-\alpha) n = k'\frac{(k-1)}{k'-1} + \frac{k'-k}{k'-1} = k.$

$$\gamma(k) \le \alpha\gamma(k') + (1-\alpha)\gamma(1) = \alpha\gamma(k') < \gamma(k)$$

$$< \frac{k}{k'}\gamma(k') \quad \Rightarrow \quad \frac{\gamma(k)}{k} < \frac{\gamma(k')}{k'}$$

(since $\gamma(k') > 0$) $\quad \Rightarrow$ Intermittent on $[k_0, \infty)$

SIGNIFICANCE OF INTERMITTENCY

$m \in [2, \infty)$, $\delta \in (0,1)$

$$P(\psi_t(x) \geq \delta \|\psi_t(x)\|_2)$$

PALEY-ZYGMUND: $X \geq 0$, $X \in L^n(\Omega)$ $P(X>0) > 0$

$n > m$ $\quad P(X > \delta \|X\|_m) \geq \dfrac{\left((1-\delta)^m\right)^{\frac{n}{n-m}} \left(E[X^m]\right)^{n/n-m}}{\left(E[X^n]\right)^{m/n-m}}$

$$\geq \frac{\left((1-\delta)^m\right)^{\frac{m}{n-m}} \left[E\,\psi_t(0)^m\right]^{n/n-m}}{E[\psi_t(0)^n]^{m/n-m}}$$

$$= c(\delta, n, m) \exp\left(t\left(\frac{n\gamma(m) - m\gamma(n)}{n-m}\right) + o(t)\right)$$

$$\varliminf \frac{1}{t} \log P(\psi_t(x) \geq \delta \|\psi_t(x)\|_2) \geq \frac{n\gamma(m) - m\gamma(n)}{n-m} \quad \forall\ n \geq m.$$

$$\geq -\inf_{n>m} \left(\frac{m\gamma(n) - n\gamma(m)}{n-m}\right) = \mu(m) \quad -(*)$$

Assume $\{\psi_j(l)\}_{l \geq 1}$ i.i.d $j \geq 1$.

$$P\left(\max_{1 \leq l \leq \exp(\alpha N)} \psi_N(l) \leq \delta \|\psi_N(0)\|_m\right)$$

$$\overset{\text{i.i.d}}{\leq} \left(1 - P(\psi_N(0) \geq \delta \|\psi_N(0)\|_m)\right)^{\lfloor \exp(\alpha N) \rfloor}$$

By $(*)$ $\leq \left(1 - e^{-N\mu_m + o(N)}\right)^{\lfloor \exp(\alpha N) \rfloor}$

$\alpha > \mu(m)$ $\quad \sum_N P\left(\max_{1 \leq l \leq \exp(\alpha N)} \psi_N(l) \leq \delta \|\psi_N(0)\|_\infty\right) < \infty$

Borel-Cantelli $\quad \lim_{N \to \infty} \frac{1}{N} \max_{1 \leq l \leq \exp(\alpha N)} \log \psi_N(x) \geq \frac{\gamma(m)}{m}$

~~Chebyshev + Union~~

$$\cancel{P\left(\max_{1\le l\le \exp(\theta N)} \psi_N(l) \geq \delta \|\psi_N(0)\|_M\right) \leq \sum_{l=1}^{\exp(\theta N)} P(\psi_N(l) \geq \delta \ldots}$$

CHEBYSHEV + UNION

$$P\left(\max_{1\le l\le \exp(\theta N)} \psi_N(l) \geq \delta \|\psi_N(0)\|_M\right) \leq \sum_{l=1}^{\exp(\theta N)} P(\psi_N(l) \geq \delta \|\psi_N(0)\|_m)$$

$$\leq (1+e^{\theta n}) \frac{\mathbb{E}\,\psi_N(0)}{\delta \|\psi_N(0)\|_m}$$

$$= e^{-N\left(\frac{\gamma(m)}{m} - \theta\right) + o(N)}$$

if $\theta < \frac{\gamma(m)}{m} \Rightarrow \overline{\lim}_{N\to\infty} \frac{1}{N} \max_{1\le l\le \exp(\theta N)} \log \psi_N(l) < \frac{\gamma(m)}{m}$

$\exists\ \theta_j \quad \theta_j < \theta_{j+1}$

$$0 < \overline{\lim}_{N\to\infty} \frac{1}{N} \max_{1\le l\le \exp(\theta_j N)} \log \psi_n(l) < \lim_{N\to\infty} \max_{1\le l\le \exp(\theta_{j+1} N)} \log \psi_N(l) < \infty$$

if $\frac{\gamma(m)}{m}$ is strictly ~~de~~increasing.

N large "$x \to \psi_N(x)$"
increasing large peaks on exp. scale.