DEFNS: $\{\Phi(t,x)\}_{t\geq 0, x\in\mathbb{R}}$ - random field

- Φ - adapted if $\Phi(t,x)$ is $\mathcal{F}_t$ m'ble $\forall x\in\mathbb{R}$
- Φ is cts in $L^2(\Omega,\mathcal{F},\mathbb{P})$ if $\lim_{n\to\infty} \sup_{\substack{s,t\in[0,N] \\ |s-t|\vee|x-y|<1/n}} \mathbb{E}[|\Phi(s,y)-\Phi(t,x)|^2] = 0$ $\forall N>0$.

PROPN: Φ adapted & continuous $\ni \|\Phi\|_{\beta,2} < \infty$ for some $\beta>0$

then $\Phi \in \bigcap_{\alpha>\beta} \mathcal{L}^{\alpha,2}$.

LECTURE - 5.

Recall: Stochastic Integral

$\xi(dy,ds)$ - spacetime white noise.

$h\in L^1(\mathbb{R}_+\times\mathbb{R})$ - $X_t(h) = \int_0^t\int_{\mathbb{R}} h(s,y)\,\xi(dy,ds)$

$\mathcal{F}_t(h) = \bigcap_{s>t} \sigma(X_r(h): r\leq s)$, $\mathcal{F}_t = \sigma\{\mathcal{F}_t(h): h\in L^1(\mathbb{R}_+\times\mathbb{R})\}$

Brownian filtration.

$\Phi = \{\Phi(t,x)\}_{t\geq 0, x\in\mathbb{R}}$

Φ (Elementary) $\Phi(t,x) = X\,\mathbb{1}_{(a,b]}(t)\,\phi(x)$, ϕ bdd m'ble on $\mathbb{R}$, X - $\mathcal{F}_a$ m'ble $L^2(\Omega)$

$$\int h\Phi\,d\xi = X\int_a^b\int_{\mathbb{R}} h(s,y)\,\phi(y)\,\xi(dy,ds)$$

Extended to all $\Phi\in\mathcal{L}^{\beta,2}$ & h satisfying $(*)_\beta$ by approximation.

MULTIPLICATIVE CASE: HEAT EQN:

$$\boxed{\circledast} \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u + \sigma(u)\xi \\ u(0,x) = \psi(x), \quad x \in \mathbb{R}, t \geq 0. \end{cases}$$

- Mild sol^n; Program; Theorem & proof

<u>Formal</u> $\phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, suppose u is a sol^n to $\circledast$.

$$\int_{\mathbb{R}} [u(t,x)\phi(t,x) - u(0,x)\phi(0,x)]dx$$

$$= \int_{\mathbb{R}} \left[\int_0^t \frac{\partial}{\partial s}[u(s,x)\phi(s,x)]\, ds\right] dx$$

$$= \int_{\mathbb{R}} \left[\int_0^t \frac{\partial u}{\partial s}\phi\, ds + \int_0^t u\frac{\partial \phi}{\partial s}\, ds\right] dx$$

$$= \int_0^t \int_{\mathbb{R}} \frac{1}{2}\Delta u\, \phi\, dx\, ds + \int_0^t \int_{\mathbb{R}} \sigma(u(s,x))\phi(s,x)\xi(dx,ds)$$

$$+ \int_0^t \int_{\mathbb{R}} u \frac{\partial \phi}{\partial s}\, ds\, dx$$

$$= \frac{1}{2}\int_0^t \int_{\mathbb{R}} u\, \Delta\phi\, dx\, ds + \int_0^t \int_{\mathbb{R}} \sigma(u(s,x))\phi(s,x)\xi(dx,ds)$$

$$+ \int_0^t \int_{\mathbb{R}} u \frac{\partial \phi}{\partial s}\, ds\, dx$$

<u>DEF:</u> $u: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a sol^n to $\circledast$ if $\forall\ \phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$

$$\int_{\mathbb{R}} u(t,x)\phi(t,x)\,dx = \int_{\mathbb{R}} \psi(x)\phi(0,x)\,dx$$

$$+ \int_0^t \int_{\mathbb{R}} u\left[\frac{1}{2}\Delta\phi + \frac{\partial\phi}{\partial s}\right] dx\, ds + \int_0^t \int_{\mathbb{R}} \sigma(u(s,x))\phi(s,x)\, \xi(dx,ds)$$

$t > 0$ fixed:

$$\phi(s,x) = \begin{cases} P_{t-s} * \eta(x) & s < t \\ \eta(x) & s = t \end{cases} \qquad \eta \in C_0^{\infty}(\mathbb{R})$$

$$\frac{\partial \phi}{\partial s} = -\frac{1}{2}\Delta\phi \quad , 0 \leq s < t$$

Apply in ⊛ , $u: \mathbb{R}_+ \times \mathbb{R}$ is a solⁿ

$$(**) \quad \int u(t,x)\eta(x)dx = \int \psi(x) P_t * \eta(x) dx + 0 + \int_0^t \int_{\mathbb{R}} \sigma(s,x) P_{t-s}(\eta(x)) \xi(dx,ds)$$

$\eta^{(n)}(x) = p_{1/n}(x_0 - x)$ for some $x_0 \in \mathbb{R}$.

$$\int u(t,x) p_{1/n}(x_0 - x) dx = \int \psi(x) P_t * p_{1/n}(x_0-x) dx + \int_0^t \int_{\mathbb{R}} \sigma(u(s,u)) P_{t-s} * p_{1/n}(x_0-x) \xi(dx,ds)$$

$\forall n \geq 1$,

Let $n \to \infty$ & to obtain

$$(\dagger) \quad u(t,x_0) = P_t * \psi(x_0) + \int_0^t \int_{\mathbb{R}} \sigma(u(s,x)) P_{t-s}(x_0 - x)\xi(dx,ds) \qquad t \geq 0, \; x_0 \in \mathbb{R}.$$

↓

Mild solution - Equivalent to the earlier form of solⁿ.

Questions : Existence, Uniqueness, Properties.

<u>Program</u>: $|\sigma(x) - \sigma(y)| \leq \lambda|x-y|$, $|\sigma(x)| \leq 1 + |x|$ $\forall x,y \in \mathbb{R}$.

<u>Existence</u>: $u^{(0)}(t,x) = \psi(x)$, $\forall t \geq 0$ ψ bdd m'ble & deterministic

$n \geq 1$ $\quad u^{(n+1)}(t,x) = P_t * \psi(x) + \int_0^t \int_{\mathbb{R}} \sigma(u^{(n)}(s,y)) P_{t-s}(x-y) \xi(dy,ds)$

Integrals are well-defined $\forall n \geq 1$.

$$d^n(t,x) = u^{(n+1)}(t,x) - u^{(n)}(t,x) \qquad n \geq 1$$

$$= \int_0^t \int_{\mathbb{R}} \left[\sigma(u^{(n)}(s,y)) - \sigma(u^{(n-1)}(s,y))\right] p_{t-s}(x-y)\, \xi(dy,ds)$$

$$H_n^2(t) = \sup_{0\leq s\leq t} \sup_{x\in\mathbb{R}} \mathbb{E}[d^n(s,x)]^2$$

$$\Rightarrow \quad \mathbb{E}[d^n(t,x)]^2 = \int_0^t \int_{\mathbb{R}} p_{t-s}^2(x-y)\, \mathbb{E}\left[\left(\sigma(u^{(n)}(s,y)) - \sigma(u^{(n-1)}(s,y))\right)^2\right] dy\, ds$$

$$\leq \lambda \int_0^t \int_{\mathbb{R}} p_{t-s}^2(x-y)\, \mathbb{E}\left[u^{(n)}(s,y) - u^{(n-1)}(s,y)\right]^2 dy\, ds$$

$$\leq \lambda \int_0^t \int_{\mathbb{R}} p_{t-s}^2(x-y)\, \mathbb{E}\left[d^{n-1}(s,y)^2\right] dy\, ds$$

$$\Rightarrow \quad H_n^2(t) \leq C_1 \int_0^1 H_{n-1}^2(s) \frac{ds}{\sqrt{4\pi(t-s)}}$$

[Generalized Gronwall] $\sum_{n=1}^{\infty} H_n(t) < \infty \Rightarrow u^{(n)}(t,x) \to u(t,x)$ in L^2

Show $\lim_{n\to\infty} \int_0^t \int_{\mathbb{R}} \sigma(u^n(s,y))\, p_{t-s}(x-y)\, \xi(dy,ds)$

$$\overset{(L^2)}{=} \int_0^t \int_{\mathbb{R}} \sigma(u(s,y))\, p_{t-s}(x-y)\, \xi(dy,ds)$$

$\Rightarrow$ Existence of $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$

$$u(t,x) \underset{L^2}{=} P_t * \varphi(x) + \int_0^t \int_{\mathbb{R}} \sigma(u(s,y))\, p_{t-s}(x-y)\, \xi(dy,ds)$$

Uniqueness: take two sol's u & v $\quad \bar{d}(t,x) = u(t,x) - v(t,x)$

THEOREM: $\exists\, \lambda_1, \lambda_2 > 0$ $|\sigma(x) - \sigma(y)| \le \lambda_1 |x-y|$ (Loc. Lipschitz), $|\sigma(x)| \le \lambda_2(1+|x|)$ (Global linear), $x, y \in \mathbb{R}$.

$\psi : \mathbb{R} \to \mathbb{R}$ - non-random & bounded m'ble.

$\exists\, \beta > 0$ & $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ (random field) in $\mathcal{L}^{\beta, 2}$

such that u satisfies (f)

- u is continuous
- u is a.s. unique among all random fields

such that $\sup_{x \in \mathbb{R}} \mathbb{E}|u(t,x)|^k \le L^k e^{Lk^3 t}$ — (1).

$\forall\, k \in [1, \infty)$, $t \ge 0$ $L = L(\lambda) > 0$.

Remarks: • $\sigma(x) = x$ equality holds in (1).

• Proof will show $\sup_{\substack{0 \le s,t \le T \\ |s-t| \le \delta}} \sup_{\substack{0 \le x,y \le M \\ |x-y| \le \eta}} |u(s,y) - u(t,x)| \le O(\delta^{1/4 - \varepsilon} + \eta^{1/2 - \varepsilon})$ as $\delta, \eta \to 0$.

Proof: $u^0(t,x) = \psi(x)$ $\forall\, t \ge 0$

Define $u^{(n+1)}$ recursively as before.

(I) $\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\, \sigma(u^{(n)}(s,y))\, \xi(dy, ds)$ $n \ge 0$

$\{\Phi(t,x)\}_{t \ge 0, x \in \mathbb{R}} \in \mathcal{L}^{\beta, 2}$ for some $\beta > 0$

$p_{t-\cdot}(x - \cdot) \in L^2(\mathbb{R}_+ \times \mathbb{R})$ $P \circledast \Phi(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\, \Phi(s,y)\, \xi(dy, ds)$

PROPN: $\Phi \in \mathcal{L}^{\beta, 2} \Rightarrow P \circledast \Phi$ has a continuous version in $\bigcap_{\alpha > \beta} L^{\alpha, 2}$.

STEPS IN PROOF:

(1) - $P * \Phi(t,x)$ is $\mathcal{F}_t$ - measurable.

(2) - $P * \Phi(t,x)$ is continuous in $L^2(\Omega)$

(3) - $P * \Phi \in \mathcal{L}^{\beta,2}$

Last ~~proposition~~ proposition of Lecture 4 $\Rightarrow$ Proposition 0.

II $\exists\ L_1, L_2$ such that $u^{(n)} \in \mathcal{L}^{\beta,2}$, $\beta = 4L_2$

$$\sup_{x\in\mathbb{R}} \mathbb{E}\left|u^{(n)}(t,x)\right|^k \le L_1^k\, e^{L_2 k^3 t}, \quad \forall\, k \ge 1,\ t>0,$$

Proof by induction.

III $\{u^{(n)}(t,x)\}_{t\ge 1,\, x\in\mathbb{R}}$ has a continuous version, $\forall n \ge 1$

Proof by Propn. 0.

BDG - Inequality Fix $k \ge 2$,

$$\left[\mathbb{E}\left|\int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\left[\sigma(u^{(n)}(s,y)) - \sigma(u^{(n-1)}(s,y))\right]\xi(dy,ds)\right|^k\right]^{2/k}$$

$$\underset{\text{(to be proved)}}{\le} C_1 k \int_0^t\int_{\mathbb{R}} p^2_{t-s}(x-y)\left(\mathbb{E}\left[\sigma(u^{(n)}(s,y)) - \sigma(u^{(n-1)}(s,y))\right]^k\right)^{2/k} dy\, ds.$$

$$\le C_2(\lambda) k \int_0^t\int_{\mathbb{R}} p^2_{t-s}(x-y)\left[\mathbb{E}\left|u^{(n)}(s,y) - u^{(n-1)}(s,y)\right|^k\right]^{2/k}$$ (*)

DEF: $$\|\Phi\|_{\beta,k} = \sup_{t\ge 0}\sup_{x\in\mathbb{R}} e^{-\beta t}\, \mathbb{E}\left[|\Phi(t,x)|^k\right]^{1/k}$$

(*) $$\le C_2(\lambda) k\, \|u^n - u^{(n-1)}\|^2_{\beta,k} \int_0^t ds\, e^{2\beta s}\int_{\mathbb{R}} p^2_{t-s}(x-y)\, dy$$

Back to the proof of Thm

BDG $\Rightarrow$ $\| u^{n+1}(t,x) - u^n(t,x) \|_k^2 \le k\, C_2(\lambda) \| u^n - u^{n-1} \|^2_{\beta,k} \frac{I_t}{e^{2\beta t}}$

$$I_t = \int_0^t ds\, e^{2\beta s} \int_{\mathbb{R}} P^2_{t-s}(x-y)\,dy \le \int_0^t ds \frac{e^{2\beta s}}{\sqrt{4\pi(t-s)}} = e^{2\beta t} \int_0^t \frac{ds\, e^{-2\beta(t-s)}}{\sqrt{4\pi(t-s)}}$$

$$\le \frac{C_3 e^{2\beta t}}{\beta}.$$

$$\| u^{(n+1)}(t,x) - u^n(t,x) \|_{\beta,k} \le \frac{\sqrt{k}\, C_1(\lambda)}{(\beta)^{1/4}} \| u^n - u^{n-1} \|_{\beta,k}$$

$\beta = M k^2$ for large M $\Rightarrow$ $\| u^{(n+1)}(t,x) - u^n(t,x) \|_{\beta,k} \le \frac{1}{2} \| u^{(n)} - u^{(n-1)} \|_{\beta,k}$

$\Rightarrow$ $\| u^{(n+1)} - u^{(n)} \|_{\beta,k} \le \frac{1}{2^n}$. $\forall\, n \ge 1$

$\hookrightarrow$ [as $\| u_1 \|_{\beta,k} \le 1/2$]

$\Rightarrow$ $u^{(n)} \to u$ in $\| \cdot \|_{\beta,k}$ and a.s. for each (t,x).

$\boxed{k=2}$ $\beta = M 2^2 = 4 L_2$ $u \in \mathcal{L}^{\beta,2}$.

- $S_t(x) = \lim_{n\to\infty} \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\sigma(u^{(n)}(s,y))\, \xi(dy,ds)$

  exists in $\| \cdot \|_{\beta,2}$ & a.s.

- $u \in \mathcal{L}^{\beta,2}$ $\bar{S}_t(x) = \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\, \sigma(u(s,y))\, \xi(dy,ds)$

  is well-defined & $\| \bar{S} - S \|_{\beta,k} = 0$ for some large k.