22/08/16

LECTURE - 4

WAVE EQUATION WITH NOISE:

$$\frac{\partial^2 v}{\partial t^2} = \Delta v + \xi \qquad 0<t<T, \ x \in \mathbb{R}$$

$(+)$
$$v(x,0) = 0 \qquad x \in \mathbb{R}$$
$$\frac{\partial v}{\partial t}(x,0) = 0$$

WEAK SOL^N: (Walsh notes). $v \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$ & $\forall \phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ such that $\phi_t(x,T) = \phi(x,T) = 0$.

$$\int_0^T \int_{\mathbb{R}} v(t,x) \left[\phi_{tt} - \phi_{xx}\right] dx\, dt = \int_0^T \int_{\mathbb{R}} \phi \, \xi(dy, ds)$$

THEOREM: There exists a unique weak solution to $(+)$ (which is continuous), namely

$$v(t,x) = \frac{1}{2} \hat{W}\left(\frac{t-x}{\sqrt{2}}, \frac{t+x}{\sqrt{2}}\right) \text{ where } \hat{W} \text{ is}$$

the modified Brownian sheet.

Proof: Same as heat eqn.

AIM:
$$\frac{\partial u}{\partial t} = \Delta u + b(u) + \sigma(u)\xi \qquad , t \geq 0, \ x \in \mathbb{R}$$

$\sigma: \mathbb{R} \to \mathbb{R}$, $b: \mathbb{R} \to \mathbb{R}$ ξ - space-time white noise.

$$u(0,x) = \psi(x).$$

Assumptions on σ & b $\boxed{?}$

DUHAMEL'S PRINCIPLE: $u(t,x) = P_t * \psi(x) + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\, b(u(s,y))\, dy\, ds$

$(*)$
$$+ \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\, \sigma(u(s,y))\, \xi(dy, ds)$$

Issues to care of : "Stochastic Integral".

- Define weak sol^n to multiplicative case.
- weak sol^n exists & unique given by (*)

WALSH: STOCHASTIC INTEGRAL:

$\xi(dy, ds)$ - space-time white noise.

$\int h\, d\xi$, $h \in L^2(\mathbb{R}^m)$ simple fns & limits

$\mathbb{E}[(\int h\, d\xi)^2] = \int h^2\, dx$.

Qn: Φ - random process on a $\{\Phi(t,x)\}_{t \geq 0, x \in \mathbb{R}}$ is a random field. what is $\int h \Phi\, d\xi$, $h \in \mathbb{L}^2(\mathbb{R}_+ \times \mathbb{R})$?

~~brew~~ BROWNIAN FILTRATION: $(\Omega, \mathcal{F}, \mathbb{P})$ ξ

$h \in L^2(\mathbb{R}_+ \times \mathbb{R})$ $\quad X_t(h) = \int_0^t \int_{\mathbb{R}} h(s, y)\, \xi(dy, ds)$

$\{X_t(h)\}_{t \geq 0}$ BM with variance $\int_0^t \int_{\mathbb{R}} h^2(s,y)\, dy\, ds$.

$$\mathbb{E}[X_{t+\epsilon}(h) - X_t(h)]^2 = \mathbb{E}\Big[\int_0^{t+\epsilon}\int_{\mathbb{R}} h(s,y)\,\xi(dy,ds) - \int_0^t h(s,y)\,\xi(dy,ds)\Big]$$

$$= \int_t^{t+\epsilon}\int_{\mathbb{R}} h^2(s,y)\, dy\, ds$$

$$\tau(t) = \inf\Big\{ s \geq 0 : \int_0^s \int_{\mathbb{R}} h^2(s,y)\, dy\, ds \geq t \Big\}$$

$B_t = X_{\tau(t)}(h)$ is a SBM.

DEFN: (B. Filt.) $\quad \mathcal{F}_t = \sigma\{ \mathcal{F}_t(h) \mid h \in L^2(\mathbb{R}_+ \times \mathbb{R}) \}$

where $\mathcal{F}_t(h) = \bigcap_{s > t} \sigma(X_r(h) : r \leq s)$

STOCHASTIC INTEGRAL: $\int h \Phi \, d\xi$ $\{\Phi(t,x)\}_{t\geq 0, x\in\mathbb{R}}$ "random field"

$h \in L^2(\mathbb{R}_+ \times \mathbb{R})$.

Elementary: $\{\Phi\}$

$X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ is $\mathcal{F}_a$-m'ble $a \geq 0$

$\Phi(t,x) = X \, \mathbb{1}_{(a,b]}(t) \, \varphi(x)$ where $\varphi : \mathbb{R} \to \mathbb{R}$ bdd m'ble.

$h \in L^2(\mathbb{R}_+ \times \mathbb{R})$
$$\int h \Phi \, d\xi = X \int_{\mathbb{R}_+}\int_{\mathbb{R}} \mathbb{1}_{(a,b]} \varphi(y) \, h(s,y) \, \xi(dy, ds)$$

Simple Random fields: Φ

$\Phi = \sum_{i=1}^{n} \Phi^{(i)}$ where $\Phi^{(i)}$ are elementary random fields and have disjoint supports $[\Phi^{(i)}\Phi^{(j)} = 0 \; ; i \neq j]$

$$\int h \Phi \, d\xi := \sum_{i=1}^{n} \int h \, \Phi^{(i)} d\xi$$

Note: Φ - Simple $\mathbb{E}\left[\int h \Phi \, d\xi\right] = 0 \quad \forall h \in L^2(\mathbb{R}_+ \times \mathbb{R})$

Φ - elementary $\mathbb{E}\left[\int h \Phi \, d\xi\right]^2 = \mathbb{E}\left[X^2 \left[\iint_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{(a,b]}(s) \, \varphi(y) \, h(s,y) \, \xi(dy,ds)\right]^2\right]$

$$= \mathbb{E}\left[X^2 \left[X_b(\varphi h) - X_a(\varphi h)\right]^2\right]$$

$X \in \mathcal{F}_a$

$$= \mathbb{E}\left[X^2 \, \mathbb{E}\left[\left(X_b(\varphi h) - X_a(\varphi h)\right)^2 \middle| \mathcal{F}_a\right]\right]$$ (by ~~Martingale prop. of X_b~~)

indep. of white noise
$$= \mathbb{E}[X^2] \, \mathbb{E}\left[\left(X_b(\varphi h) - X_a(\varphi h)\right)^2\right]$$

$$= \mathbb{E}[X^2] \int_a^b \int_{\mathbb{R}} h^2(s,y) \, \varphi^2(y) \, dy \, ds$$

$$= \iint_{\mathbb{R}_+ \, \mathbb{R}} h^2(s,y) \, \mathbb{E}\left[\Phi(s,y)^2\right] dy \, ds$$

$$\Rightarrow \mathbb{E}\left[\left(\int h \Phi \, d\xi\right)^2\right] = \int_{\mathbb{R}_+} \int_{\mathbb{R}} h^2(s,y) \, \mathbb{E}\left[\Phi(s,y)^2\right] dy \, ds$$

$\downarrow$

Walsh Isometry. $\forall \Phi$ simple.

"Correct" Limit space: $N_{\beta,2}(\Phi) = \sup_{t \geq 0} \sup_{x \in \mathbb{R}} e^{-\beta t} \|\Phi(t,x)\|_2$

$\|\Phi(t,x)\|_2 = \left(\mathbb{E}\, \Phi(t,x)^2\right)^{1/2}$ (norm- upto modification on space of random fields)

$$N_{\beta,2} = \{ v : \underbrace{N_{\beta,2}(v)}_{\|v\|_{\beta,2}} < \infty \}$$

$\|\Phi\|_{\beta,2} < \infty$ for some $\beta > 0$,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} h^2(s,y) \, \mathbb{E}\left[\Phi^2(s,y)\right] dy\, ds \leq \int_{\mathbb{R}_+} \int_{\mathbb{R}} h^2(s,y) \, e^{2\beta s} \|\Phi\|_{\beta,2}^2 \ldots$$

$$= \|\Phi\|_{\beta,2}^2 \int_{\mathbb{R}_+} \int_{\mathbb{R}} h^2(s,y) \, e^{2\beta s} dy \, ds$$

Φ simple $\Rightarrow \|\Phi\|_{\beta,2} < \infty \quad \forall \beta > 0$.

$$\mathcal{L}^{\beta,2} = \overline{\{ \Phi : \Phi \text{ simple random field} \}}^{\,\|\cdot\|_{\beta,2}}$$

$h \in L^2(\mathbb{R}_+ \times \mathbb{R})$ for $\beta > 0$, $\int_{\mathbb{R}_+} \int_{\mathbb{R}} e^{2\beta s} h^2(s,y) \, dy \, ds < \infty$ for some β

$\circledast_\beta$

Stochastic integral for $\Phi \in \mathcal{L}^{\beta,2}$ & $h \in \circledast_\beta$. $\Phi \in \mathcal{L}^{\beta,2}$

$\exists\ \Phi^{(n)}$ - simple $\quad \|\Phi^{(n)} - \Phi\|_{\beta,2} \to 0$ as $n \to \infty$.

$\Rightarrow \Phi^{(n)}$ is a Cauchy sequence.

$$\mathbb{E}\left|\int h\,\Phi^{(n)} d\xi - \int h\,\Phi^{(m)} d\xi\right|^2 \leq \|\Phi^{(n)} - \Phi^{(m)}\|^2_{\beta,2} \iint e^{2\beta s} h^2(s,y)\,dy\,ds.$$

$\Rightarrow \left\{\int h\,\Phi^{(n)} d\xi\right\}_{n\geq 1}$ is Cauchy in $L^2(\Omega, \mathcal{F}, P)$ and hence

$$\int h\,\Phi\, d\xi \overset{L^2}{=} \lim_{n\to\infty} \int h\,\Phi^{(n)} d\xi. \text{ – Independent of choice of } \Phi^{(n)}.$$

PROPERTIES ~~Again~~

$$\mathbb{E}\left[\int h\,\Phi\, d\xi\right] = 0$$

$$\mathbb{E}\left[\left(\int h\,\Phi\, d\xi\right)^2\right] = \int_{\mathbb{R}_+}\int_{\mathbb{R}} h^2(s,y)\,\mathbb{E}\left[\Phi(s,y)^2\right] dy\,ds$$

$\downarrow$

Walsh Isometry holds.

$h \to \int h\,\Phi\, d\xi$ is linear over h satisfying $\circledast_\beta$.

$\Phi \to \int h\,\Phi\, d\xi$ is linear a.s. for any h in $\circledast_\beta$ $\forall\ \Phi \in \mathcal{L}^{\beta,2}$.

PROPOSITION: $t \geq 0$ $h \in L^2([0,t]\times\mathbb{R})$ $\Phi \in \mathcal{L}^{\beta,2}$ for some $\beta > 0$

$$M_t = \int_0^t \int_{\mathbb{R}} h(s,y)\,\Phi(s,y)\,\xi(dy,ds)$$

defines a Martingale in $L^2(\Omega, \mathcal{F}, P)$ w.r.t $\mathcal{F}_t$

$$\langle M_t\rangle = \int_0^t \int_{\mathbb{R}} h^2(s,y)\,\Phi^2(s,y)\,dy\,ds$$

Proof sketch: Φ Elementary. $M_t = X\left[X_{b\wedge t}(h\varphi) - X_{a\wedge t}(h\varphi)\right]$

Show for elementary, simple & then limits. ∎

$\mathcal{L}^{\beta,2}$: EXAMPLES: (1) Φ simple. $\Phi \in \mathcal{L}^{\beta,2}$. (2) Φ $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ deterministic m'ble $\|\Phi\|_{\beta,2} < \infty \Rightarrow \Phi \in \mathcal{L}^{\beta,2}$

DEFNS: $\{\Phi(t,x)\}_{t\geq 0, x\in\mathbb{R}}$ - random field

- Φ - adapted if $\Phi(t,x)$ is $\mathcal{F}_t$ m'ble $\forall x \in \mathbb{R}$
- Φ is cts in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ if $\lim_{n\to\infty} \sup_{\substack{s,t\in[0,N] \\ |s-t| \vee |x-y| < 1/n}} \mathbb{E}[|\Phi(s,y)-\Phi(t,x)|^2] = 0$

$\forall N > 0.$

PROPN: Φ adapted & continuous $\ni \|\Phi\|_{\beta,2} < \infty$ for some $\beta > 0$

then $\Phi \in \bigcap_{\alpha>\beta} \mathcal{L}^{\alpha,2}$.