$$\hat{V}(u,v) = \frac{1}{2}\int_0^u \int_{-v}^{v} \xi(d\alpha, d\beta) = \frac{1}{2}\xi(R_{u,v}) = \frac{1}{2}\hat{W}_{u,v}.$$

08/08/16

LECTURE - 3

Recall: $\xi(t,x)$ - $t\geq 0$, $\mathbb{R}\ni x$, space-time white noise

- GRF on $h(\mathbb{R}_+ \times \mathbb{R})$ ↳ finite Lebesgue measure Borel sets.

- ξ is not a measure but $L^2(\Omega, \mathcal{F}, \mathbb{P})$-valued σ-finite measure.

$(\int h\, d\xi)_{h\in L^2(\mathbb{R}_+\times\mathbb{R})}$ - GRF with $\mathrm{Cov}(\int h_1 d\xi, \int h_2 d\xi) = \int h_1(x) h_2(x)\, dx$.

$$\mathbb{E}\left| \xi\left(\bigcup_{i=1}^{\infty} A_i\right) - \sum_{i=1}^{\infty} \xi(A_i)\right|^2 = 0 \quad A_i \text{ disjoint}.$$

STOCHASTIC LINEAR HEAT EQUATION:

Give meaning to (*)
$$\begin{cases} \dfrac{dv}{dt} = \dfrac{1}{2}\Delta v + \xi(dt\,dx), & t>0,\ x\in\mathbb{R} \\ v(0,x) = \psi(x) \end{cases}$$

DEFN (WEAK SOLN TO *): $u: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $[u \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})]$ is a weak solution to (*), if u is (1) locally integrable &

(1)
$$-\iint_{\mathbb{R}_+\mathbb{R}} u(t,x)\frac{d\varphi}{dt}(t,x)\,dt\,dx \overset{L^2 \text{ sense}}{=} \frac{1}{2}\iint_{\mathbb{R}_+\times\mathbb{R}} u(t,x)\Delta\varphi(t,x)\,dx\,dt + \iint_{\mathbb{R}_+\times\mathbb{R}} \varphi(t,x)\,\xi(dt,dx)$$

$\forall\ \varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$

(2)
$$\lim_{t\to 0} \int u(t,x)\varphi(x)\,dx \overset{L^2\text{-sense}}{=} \int_{\mathbb{R}} \psi(x)\varphi(x)\,dx \quad \forall\ \varphi \in C_c^\infty(\mathbb{R})$$

we'll work towards a uniqueness & existence theorem to (*).

- HEAT EQUATION FACTS :

$$\mathcal{H}f = \frac{\partial f}{\partial t} - \frac{1}{2}\Delta f.$$

ADJOINT $\mathcal{H}^* f = -\frac{\partial f}{\partial t} - \frac{1}{2}\Delta f.$

$$\int \mathcal{H}f, g \, dt\, dx = \int f \mathcal{H}^* g \, dt\, dx \qquad \forall f, g \in C_c^\infty(\mathbb{R}_* \times \mathbb{R}).$$

HEAT SEMI-GROUP : $x, y \in \mathbb{R}$. $t > s \geq 0$

$$p(t,x;s,y) = \frac{e^{-\frac{(x-y)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}$$

(i) Fix s, y. $g(t,x) = p(t,x;s,y)$ $t \geq s$, $y \in \mathbb{R}$.

$$\frac{\partial g}{\partial t} = \frac{1}{\sqrt{2\pi}}\left[-\frac{1}{2}(t-s)^{-3/2} e^{-\frac{(x-y)^2}{2(t-s)}} + \frac{1}{\sqrt{t-s}} e^{-\frac{(x-y)^2}{2(t-s)}} \frac{(x-y)^2}{2(t-s)^2}\right]$$

$$\frac{\partial g}{\partial x} = \frac{1}{\sqrt{2\pi}}\left[\frac{-(x-y)}{(t-s)^{3/2}} e^{-\frac{(x-y)^2}{2(t-s)}}\right]$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{1}{\sqrt{2\pi}}\left[\frac{-e^{-\frac{(x-y)^2}{2(t-s)}}}{(t-s)^{3/2}} + \frac{(x-y)^2 e^{-\frac{(x-y)^2}{2(t-s)}}}{(t-s)^{5/2}}\right]$$

$$\Rightarrow \frac{\partial g}{\partial t} = \frac{1}{2}\Delta g. \quad t > s, \quad x, y \in \mathbb{R}.$$

$\phi \in C_c^\infty([s,\infty) \times \mathbb{R})$

$$-\int_{\mathbb{R}_* \times \mathbb{R}} g(t,x) \frac{\partial \varphi}{\partial t}(t,x)\, dt\, dx - \frac{1}{2}\iint g \, \Delta\varphi \, dt\, dx$$

$$= -\int_{\mathbb{R}} g(t,x)\, \varphi(t,x)\Big|_s^\infty dx + \int_{\mathbb{R}}\int_{\mathbb{R}_+} \frac{\partial g}{\partial t} \varphi \, dt\, dx$$

(Integration by parts.)

$$- \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} \Delta g \, \varphi \, dt\, dx.$$

$$\Rightarrow \; -\int_{\mathbb{R}_+\times\mathbb{R}} g \frac{\partial \varphi}{\partial t}\,dt\,dx - \frac{1}{2}\int_{\mathbb{R}_+\times\mathbb{R}} g\,\Delta\varphi\,dt\,dx = \varphi(s,y) + 0$$

$$\Big[\text{since } \lim_{t\to s} \int \phi(t,x;s,y)\,\varphi(t,x)\,dx = \varphi(s,y)\Big]$$

$$\Big[\;\&\; \int_{\mathbb{R}}\int_{\mathbb{R}_+} \frac{\partial g}{\partial t}\,\varphi\,dt\,dx - \frac{1}{2}\iint \Delta g\,\varphi\,dt\,dx = 0\Big]$$

sol^n to heat eqn.

Candidate for solution to (*)

$$p(t,x;s,y) = g(t,x) \qquad p_t(x) = \frac{e^{-x^2/2s}}{\sqrt{2\pi t}}$$

$$\;\; = p_{t-s}(x-y)$$

$$u(t,x) = p_t * \psi(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\,\xi(dy,ds)$$

Think of earlier sol^n $\phi(s,y) = \int_{\mathbb{R}} \phi(t,x)\,d\mu$ $\mu = \delta_{(s,y)}$

All make sense, because $p_{t-s}(x-y) \in L^2$.
[in L^2-sense as $d\xi$ integrals are involved]

To show

$$-\int u \frac{\partial\varphi}{\partial t} - \frac{1}{2}\int u\,\Delta\varphi \overset{?}{=} \int \varphi\,d\xi$$

Fact 1: $\dfrac{\partial\, p_t * \psi(x)}{\partial t} = \dfrac{1}{2}\Delta p_t * \psi(x)$;

Integration by Parts $\Rightarrow$ $-\int_{\mathbb{R}_+\times\mathbb{R}} p_t * \psi \dfrac{\partial \varphi}{\partial t} - \dfrac{1}{2}\int_{\mathbb{R}_+\times\mathbb{R}} p_t * \psi\,\Delta\varphi = 0$ $\forall \varphi \in C_c^\infty$ (A)

By A

$$-\int u\frac{d\varphi}{dt} - \frac{1}{2}\int u\,\Delta\varphi = -\int_{\mathbb{R}_+}\int_{\mathbb{R}}\Big[\int_0^t p_{t-s}(x-y)\,\xi(dy,ds)\Big]\Big(\frac{\partial\varphi}{\partial t} + \frac{\Delta\varphi}{2}\Big)\,dt\,dx$$

Stochastic Fubini :

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \mathbb{E}\int_{\mathbb{R}_+}\int_{\mathbb{R}} f(t-s, x-y)\, \xi(dy, ds) \left(\frac{1}{2}\Delta\varphi + \frac{\partial\varphi}{dt}\right) dt\, dx$$

$$f_r(z) = \begin{cases} p_r(z) & r>0,\ z\in\mathbb{R} \\ 0 & \text{elsewhere.}\end{cases}$$

$\boxed{\text{Fubini}}$

$$\overset{?}{=} \int_{\mathbb{R}_+}\int_{\mathbb{R}} \left[\int_{\mathbb{R}_+}\int_{\mathbb{R}} f(t-s, x-y)\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}\Delta\varphi\right) dt\, dx\right] \xi(dy, ds)$$

$$= \int_{\mathbb{R}_+}\int_{\mathbb{R}} \varphi(s, y)\, \xi(ds, dy) = \int \varphi \, \xi(dy, ds)$$

Stochastic Fubini : Fubini in L^2-sense since ξ not a measure a.e.

Then, u is a weak solution.

UNIQUENESS : $H\varphi = \frac{\partial}{\partial t}\varphi - \frac{1}{2}\Delta\varphi$, $H^*\varphi = -\frac{\partial\varphi}{\partial t} - \frac{1}{2}\Delta\varphi$

There are two weak solutions : u, v

$$\int_{\mathbb{R}_+\times\mathbb{R}} u(H^*\varphi) = \int_{\mathbb{R}_+\times\mathbb{R}} \varphi\, d\xi \ ; \quad \int_{\mathbb{R}_+\times\mathbb{R}} v H^*\varphi = \int \varphi\, d\xi$$

$$\Rightarrow U = u - v \quad \Rightarrow \int_{\mathbb{R}_+\times\mathbb{R}} U H^*\varphi = 0 \quad \forall\ \varphi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R})$$

$$\Rightarrow U \equiv 0 \ \text{a.s.}$$

[Given $f\in C_c^\infty(\mathbb{R}_+\times\mathbb{R})$, $\exists\ \varphi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R})$ $H^*\varphi = f$]

then $\int_{\mathbb{R}_+\times\mathbb{R}} U H^*\varphi = 0 \Rightarrow \int Uf = 0 \ \forall\ f\in C_c^\infty(\mathbb{R}_+\times\mathbb{R})$

$$\Rightarrow U = 0.$$

[Issues with the proof !!!]

$$\mathbb{E}[u^2(t,x)] \leq 4\Big[[P_t * \psi(x)]^2 + \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\,\xi(ds,dy)\Big]^2\Big]$$

$$= 4\Big[[P_t * \psi(x)]^2 + \int_0^t \int_{\mathbb{R}} P_{t-s}^2(x-y)\,dy\,ds\Big]$$

$$\int_{\mathbb{R}} P_{t-s}(x-y)\,P_{t-s}(y-x)\,dy = P_{2(t-s)}(x,x) = \frac{1}{\sqrt{4\pi(t-s)}}$$

ψ - bounded
$$= 4[P_t * \psi(x)]^2 + \int_0^t \frac{1}{\sqrt{4\pi(t-s)}}\,ds$$

$$\sup_{x\in\mathbb{R}} \sup_{t\in[0,T]} \mathbb{E}[u^2(t,x)] < \infty.$$

<u>THM</u>: The stochastic heat equation has an a.s. unique sol^n $u: [0,T]\times\mathbb{R} \to \mathbb{R}$ that is given by
$$u(t,x) = P_t * \psi(x) + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\,\xi(dy,ds)$$
where ψ is bounded & further
$$\sup_{x\in\mathbb{R}} \sup_{0\le t\le T} \mathbb{E}[u^2(t,x)] < \infty.$$

<u>PROOF OF STOCHASTIC FUBINI</u>:

<u>LEMMA</u>: $f \in L^2(\mathbb{R}^m)$, η - finite measure on $\mathbb{R}^m$, ξ - white noise on $\mathbb{R}^m$

$$\int_{\mathbb{R}^m} \eta(dx) \int_{\mathbb{R}^m} \xi(dy)\,f(x-y) = \int \xi(dy) \int f(x-y)\,\eta(dx) \quad \text{a.s.}$$

<u>PROOF</u>: [Main Ideas:] φ differentiable on $\mathbb{R}$, $\mathrm{supp}(\varphi) \subseteq [0,1]$

$$\varphi^n(z) = \{\varphi(j/n) : j/n \le z \le \tfrac{j+1}{n}\}$$

$$\int \varphi^n(z)\,\xi(dz) = \cancel{\sum_{j=0}^{n-1} \varphi(\tfrac{j}{n})[\xi(\tfrac{j+1}{n}) - \xi(\tfrac{j}{n})]} \sum_{j=0}^{n-1} \varphi(\tfrac{j}{n})\,\xi(\tfrac{j}{n}, \tfrac{j+1}{n}]$$

$$\overset{D}{=} \sum_{j=0}^{n-1} \varphi(\tfrac{j}{n})\,[B_{\frac{j+1}{n}} - B_{\frac{j}{n}}]$$

$$= - \sum_{k=1}^{n} \frac{B_k}{n}\left[\varphi\left(\tfrac{k}{n}\right) - \varphi\left(\tfrac{k+1}{n}\right)\right] \xrightarrow{n\to\infty}$$

As $n \to \infty$ $\int \varphi(z)\, \xi(dz) = - \int_0^1 B_t\, \phi'(t)\, dt$

PROPN 2.8 (DAVAR'S BOOK):

$$\int \phi(x)\, \xi(dx) = (-1)^m \int_{\mathbb{R}^m} \frac{\partial^m \varphi(x)}{\partial x_1 \cdots \partial x_m}\, \xi(R(x))\, dx \qquad \varphi \in C_c^\infty(\mathbb{R}^m)$$

$R(x)$: appropriate rectangle.

Back to the proof

$$\int_{\mathbb{R}^m} \eta(dx) \int_{\mathbb{R}^m} \xi(dy)\, f(x-y) = (-1)^m \int_{\mathbb{R}^m} \eta(dx) \int_{\mathbb{R}^m} \frac{\partial^m f(x-y)}{\partial x_1 \cdots \partial x_m}\, \xi(R(y))\, dy$$

$$\overset{\text{Fubini}}{=} (-1)^m \int_{\mathbb{R}^m} \xi(R(y))\, dy \int_{\mathbb{R}^m} \frac{\partial^m f(x-y)}{\partial x_1 \cdots \partial x_m}\, \eta(dx)$$

$$\int_{\mathbb{R}^m} \xi(dy) \underbrace{\int_{\mathbb{R}^m} \eta(dx)\, f(x-y)}_{g(x)} \overset{\text{Prop 2.8}}{=} (-1)^m \int_{\mathbb{R}^m} \xi(R(y))\, \frac{\partial^m g(y)}{\partial y_1 \cdots \partial y_m}$$

Fubini, diff. under integral, $B(x) = \xi(R(x))$ ← continuous modification

& show for $f \in C_c^\infty(\mathbb{R}^m)$ & take limits for $f \in L^2(\cdot)$. [True for B. Sheet]

$$u(t,x) = p_t * \psi(x) + Z_t(x), \quad \text{where } Z_t(x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\, \xi(dy, ds)$$

- $x \in \mathbb{R}$, $\left\{ Z_t(x) - \left(\frac{\pi}{2}\right)^{-1/4} X_t \right\}_{t \geq 0}$ is a mean-0 Gaussian process & has a continuous version in $[0,\infty)$ & infinitely diff'ble version on $(0,\infty)$ where X_t is a FBM with $H = 1/4$.
- Fix $t > 0$, $\{Z_t(x) - B(x)\}_{x \in \mathbb{R}}$ is a GRF on $\mathbb{R}$ with a C^∞ version for $B(x)$. is a mean 0 GRF with

$$\mathbb{E}\left[(B(x) - B(y))^2\right] = |x-y|.$$