01/08/16.

LECTURE - 2.

Recall: $\dfrac{\partial u}{\partial t} = -\Delta u \qquad u(0,x) = \varphi(x), \quad x \in [0,1]$

$u(t,0) = u(t,1) = 0.$

$$(*) \begin{cases} \dfrac{\partial u}{\partial t} = -\Delta u + f(t,x). \\ \quad \hookrightarrow f(t,x) = \xi(t,x) - \text{space-time white noise. Additive noise.} \\ \dfrac{\partial u}{\partial t} = \Delta u + \sigma(u)\xi(t,x) + b(u). - \text{Multiplicative.} \end{cases}$$

$\lambda = 0$, require u to be differentiable. But $\lambda > 0$, u might be rough. Need to make sense of $(*)$.

Weak Solution formulation: $\varphi \in C_c^\infty(\mathbb{R}_* \times \mathbb{R})$ test functions.

$$-\int_0^\infty \int_{\mathbb{R}} u(t,x) \frac{\partial \varphi}{\partial t}(t,x)\, dt\, dx = \int_0^\infty \int_{\mathbb{R}} u(t,x)\, \Delta\varphi(t,x)\, dt\, dx + \underbrace{\lambda \int_0^\infty \int_{\mathbb{R}} \varphi(t,x)\, \underline{\xi(dt,dx)}_{?}}_{\text{integrals?}}$$

Multiplicative case: $\displaystyle\int_0^\infty \int_{\mathbb{R}} \sigma(u)\, \varphi(t,x)\, \xi(dt,dx)$ — stochastic Integrals?

- $u(t,x) := P_t * \phi(x)$ on $P_t(z) = \dfrac{e^{-z^2/2t}}{\sqrt{2\pi t}}$ then $\dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u, \quad u(0,x) = \phi(x).$ — $\lambda = 0$ case in $(*)$.

- $f(t,x)$ - deterministic. — $\checkmark$ Set-up.

$$u(t,x) := P_t * \phi(x) + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y)\, f(s,y)\, dy\, ds$$

then $\dfrac{du}{dt} = \Delta u + f \quad \& \quad u(0,x) = \varphi(x).$

For ⊛ - ~~rand~~ i.e., $f = \xi$, random

Then ~~solutions~~ $u(t,x) = P_t * \phi(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\xi(dy,ds)$

"STOCHASTIC CONVOLUTION" (?)

is a "solution" (?) to

" $\frac{\partial u}{\partial t} = \Delta u + \xi$, $u(0,x) = \varphi(x)$, $t>0$, $x \in \mathbb{R}$ "

CH 2. (Davar's book.) T - set, Gaussian Random field on T.

$\{X_t\}_{t\in T}$ is GRF on T if $(X_{t_1}, \ldots, X_{t_m})$ is a Gaussian r. vector for $m \geq 1$, $t_i \in T$.

$T \subseteq \mathbb{R}$ GRF = G.Proc.

<u>White-noise on $\mathbb{R}^m$</u>: $\mathcal{L}(\mathbb{R}^m) = \{B \subseteq \mathbb{R}^m \mid B$ - Borel & $|B| < \infty\}$

ξ - white noise on $\mathbb{R}^m$ if ξ is a $\mathcal{L}(\mathbb{R}^m)$ - indexed GRF with mean 0 & $\mathrm{Cov}(\xi(A), \xi(B)) = |A \cap B|$

i.e., $\{\xi(A)\}_{A \in \mathcal{L}(\mathbb{R}^m)}$ $\ni$ $\mathbb{E}\,\xi(A) = 0$, $\mathrm{cov}(\xi(A), \xi(B)) = |A \cap B|$. $\forall A, B \in \mathcal{L}(\mathbb{R}^m)$.

<u>EXISTENCE</u>: $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, $A_1, \ldots, A_n \in \mathcal{L}(\mathbb{R}^m)$

$$\sum_{i,j=1}^n \alpha_i \bar{\alpha}_j |A_i \cap A_j| = \sum_{i,j} \alpha_i \bar{\alpha}_j \int_{\mathbb{R}^m} \mathbb{1}_{A_i} \mathbb{1}_{A_j} \, dx$$

$$= \int_{\mathbb{R}^m} \sum_{i,j} \alpha_i \bar{\alpha}_j \mathbb{1}_{A_i} \mathbb{1}_{A_j} \, dx$$

$$= \int_{\mathbb{R}^m} \Big| \sum_i \alpha_i \mathbb{1}_{A_i} \Big|^2 dx \geq 0.$$

$\Rightarrow$ Cov. fn is positive semi-definite & hence white-noise exists.

<u>PROPERTIES of ξ</u>: $\xi(A) \overset{d}{=} N(0, |A|)$ $A \in \mathcal{L}(\mathbb{R}^m)$

$\xi(A)$ & $\xi(B)$ are indep if $A \cap B = \emptyset$

$A_1 \cap A_2 = \phi$ $\quad \mathbb{E}\left|\xi(A_1 \cup A_2) - \xi(A_1) - \xi(A_2)\right|^2$

$A_1, A_2 \in \mathcal{L}(\mathbb{R}^m)$.

$$= |A_1 \cup A_2| - |A_1| - |A_2| + 2|A_1 \cap A_2|$$

$$= 0 \quad \text{if } A_1 \cap A_2 = \phi.$$

ξ is finitely additive in L^2- sense.

$B_1 \supseteq B_2 \supseteq \cdots \quad \ni \bigcap_{m=1}^{\infty} B_m = \phi \quad B_i \in \mathcal{L}(\mathbb{R}^m)$

$$\mathbb{E}|\xi(B_n)|^2 = |B_n| \longrightarrow 0 \quad \text{as } n \to \infty.$$

$\Rightarrow A_i \in \mathcal{L}(\mathbb{R}^m)$, $A_i \cap A_j = \phi$ then $\mathbb{E}\left|\xi\left(\bigcup_{i=1}^{\infty} A_i\right) - \sum_{i=1}^{\infty} \xi(A_i)\right|^2 = 0$

ξ - $L^2(\Omega, \mathcal{F}, P)$ valued σ-finite measure i.e., a σ-finite measure in the L^2- sense.

[complete construction : Pg 286, Walsh Notes/Book].

Ex. $m=1$, $\sum_{j=1}^{2^n} \left|\xi\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right]\right|^2 \xrightarrow{P} 1$ but $\sum_{j=1}^{2^n} \left|\xi\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right]\right| \to \infty$ a.s.

INTEGRATION OVER ξ $\quad S(\mathbb{R}^m) = \left\{h : \mathbb{R}^m \to \mathbb{R} \;\middle|\; h(x) = \sum_{i=1}^{m} \alpha_i \mathbb{1}_{A_i}(x), \alpha_i \in \mathbb{R}, A_1, \ldots, A_n \in \mathcal{L}(\mathbb{R}^m)\right\}$

$$\int h\, d\xi = \int h(x)\, \xi(dx) = \sum_{i=1}^{n} \alpha_i\, \xi(A_i) \quad \text{for } h(x) = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}(x)$$

PROP: Definition doesn't depend on representation of h.

$$\int(\alpha h + \beta g)\, d\xi = \alpha \int h\, d\xi + \beta \int g\, d\xi \qquad h, g \in S(\mathbb{R}^m),\ \alpha, \beta \in \mathbb{R}.$$

$\left(\int h\, d\xi\right)_{h \in S(\mathbb{R}^m)}$ — GRF on $S(\mathbb{R}^m)$ with mean 0.

$\mathbb{E}\left[\int h\, d\xi\right] = 0 \qquad \mathbb{E}\left[\int h_1\, d\xi \int h_2\, d\xi\right] = \mathbb{E}\left[\sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_i \beta_j\, \mathbb{E}[\xi(A_i)\xi(B_j)]\right]$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_i \beta_j\, |A_i \cap B_j|$$

$$= \int_{\mathbb{R}} h_1(x) h_2(x)\, dx.$$

$\Rightarrow \mathbb{E}\left[\left(\int h_1 d\xi - \int h_2 d\xi\right)^2\right] = \int |h_1 - h_2|^2 dx$

$h \in L^2(\mathbb{R}^m)$, $\exists\ h_n \in S(\mathbb{R}^m) \ni h_n \to h$ in L^2.

$\Rightarrow \quad \mathbb{E}\left[\left(\int h_n d\xi - \int h_m d\xi\right)^2\right] = \int |h_n - h_m|^2 dx$

$\Rightarrow \quad \left(\int h_n d\xi\right)_{n \geq 9}$ is cauchy in $L^2(\Omega, \mathcal{F}, \mathbb{P})$

$\Rightarrow \quad \int h d\xi = \lim_{n \uparrow \infty} \int h_n d\xi$. in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Doesn't depend on choice of h.

$\int (\alpha f + \beta g) d\xi = \alpha \int h d\xi + \beta \int g d\xi$

$\left(\int h d\xi\right)_{h \in L^2(\mathbb{R}^m)}$ is a GRF on $L^2(\mathbb{R}^m)$.

$\mathbb{E}\left[\int h_1 d\xi \int h_2 d\xi\right] = \int h_1(x) h_2(x) dx$. $\quad \mathbb{E}\left[\int h d\xi\right] = 0$.

<u>ISOMETRY</u>: $\quad L^2(\mathbb{R}^m) \longrightarrow L^2(\Omega, \mathcal{F}, P)$.

$h \xrightarrow[\text{Isometry.}]{} \int h d\xi$ – Wiener Integral.

Stochastic Convolution : $x \in \mathbb{R}^m$, $f \in L^2(\mathbb{R}^m)$

$f * \xi(x) = \int f(x-y)\, \xi(dy)$

$(\omega, x) \longmapsto (f * \xi)(\omega, x)$.

$f \in C_c^\infty(\mathbb{R}^m) \quad \mathbb{E}\, |f * \xi(x) - f * \xi(z)|^2 = \mathbb{E}\left| \int f(x-y) - f(z-y))\xi(dy)\right|$

$= \int |f(x-y) - f(z-y)|^2 dy \leq c|x-z|^2$

$\Rightarrow \quad x \longmapsto \mathbb{E}\, |f * \xi(x)|^2$ is continuous in

version of Kol. cty thm $\Rightarrow \quad x \to f * \xi(x)$ cts a.s.

Measure theoretic arguments $\Rightarrow (\omega, x) \to f * \xi(x)$ is jointly measurable.

Approximate $f \in L^2(\mathbb{R}^m)$.

EXAMPLES : $\underline{m=1}$. ξ - white noise on $\mathbb{R}$. $t \in [0,\infty)$

$B_t = \xi([0,t]) \Rightarrow B_t \overset{d}{=} N(0,t)$

$\left.\begin{array}{l} \\ \mathbb{E}[B_t B_s] = s \wedge t. \end{array}\right\}$ Continuous version is the SBM.

$\underline{m=2}$. $\psi^{(t)}: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ fix $t > 0$

$\psi^{(t)}(s,x) = 1_{[0,t]}(s)\, \phi(x) \qquad \phi \in C_c^\infty(\mathbb{R})$

$\int \psi^{(t)}(s,x)\, d\xi = \int_0^t \phi(x)\, \xi(ds, dx)$

$\mathbb{E}\left[\left(\int \psi^{(t)}(s,x)\, d\xi\right)^2\right] = \int \psi^{(t)}(s,x)^2\, ds\, dx = t \int_{\mathbb{R}} \phi^2(x)\, dx$

$\downarrow$

BM with variance $t\int \phi^2 dx$.

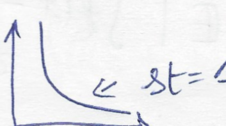
$\underline{\text{EXAMPLE 3}}$: $\mathbb{R}^n_+ = \{ t \in \mathbb{R}^n \mid t_i \geq 0 ;\ 1 \leq i \leq n \}$.

$t \in \mathbb{R}^n_+$. $W_t = W_{(t_1,\cdot\cdot, t_n)} = \xi\left(\prod_{i=1}^n [0,t_i]\right)$

$\mathbb{E}W_t = 0 \qquad \mathbb{E}W_t W_s = \prod_{i=1}^n \left|[0, t_i \wedge s_i]\right| = \prod_{i=1}^n (t_i \wedge s_i)$

Brownian sheet.

Fix $a \geq 0$; $n=2$. $\{W_{a,t}\}_{t\geq 0}$ — BM with variance at.



curve. $X_t = W_{e^t, e^{-t}}$, $t \in \mathbb{R}$.

$\mathbb{E}X_t = 0$, $\quad \mathbb{E}[X_t^2] = e^t \cdot e^{-t} = 1$.

$\mathbb{E}[X_r X_u] = e^{-|r-u|} \Rightarrow X_t$ - O-U process.

Along $s=t$, $M_t = W_{t,t} \overset{d}{=} N(0,t^2)$; Martingale.

Scaling : $\frac{1}{\alpha\beta} W_{\alpha^2 s, \beta^2 t} \overset{d}{=} W_{s,t}$

Inversion : $st\, W_{1/s, 1/t} \overset{d}{=} W_{s,t}$

Translation : $W_{a+s,b+t} - W_{a+s,t} - W_{s,b+t} + W_{a,b} \overset{d}{=} W_{s,t}$

Brownian sheet.

- (Walsh : Ch- 1) - Markov property & Continuity & LIL & stopping points.

Application : (stochastic wave equation)

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} ; \quad V(0,x) = \frac{\partial V}{\partial t}(0,x) = 0. \quad x \in \mathbb{R}, \ t > 0.$$

Forcing $\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + f(t,x)$: Sol^n $V(t,x) = \frac{1}{2} \int_0^t \int_{x+s-t}^{x+t-s} f(s,y)\, dy\, ds$

Verify directly by differentiating.

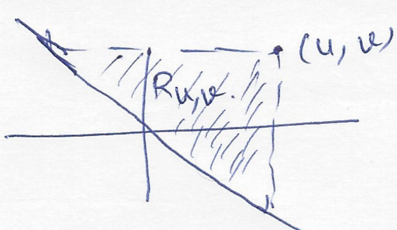
Rotate by 45° $\Rightarrow$ $u = \frac{s-y}{\sqrt{2}}$, $v = \frac{s+y}{\sqrt{2}}$ $\quad \hat{V}(u,v) = V(s,y)$.

$\hat{f}(u,v) = f(s,y)$. $\quad \hat{V}(u,v) = \frac{1}{2} \int_0^v \int_{-v}^{u} \hat{f}(\alpha,\beta)\, d\alpha\, d\beta$

STOCHASTIC WAVE EQN : $\quad x \in \mathbb{R}, t > 0$.

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + \xi(dt,dx) ; \quad V(0,x) = \frac{\partial V}{\partial t}(0,x) = 0. \quad - (**)$$

$D = \{(u,v) \mid u + v \geq 0\}$ $\quad R_{uv} = D \cap (-\infty, u] \times (-\infty, v]$



$\hat{W}_{u,v} = \xi(R_{u,v})$

$\hat{W}_{u,v}$ is not a Brownian sheet (0 on $u + v = 0$)

$\hat{W}_{u,v} - \hat{W}_{u,0} - \hat{W}_{0,v} \overset{d}{=} \tilde{W}_{u,v}$ (Brownian sheet $u, v \geq 0$).

Solution to (**) $V(t,x) = \frac{1}{2} \int_0^t \int_{x+s-t}^{x+t-s} \xi(dy,ds)$

formal

$$\hat{V}(u,v) = \frac{1}{2}\int_0^v \int_{-v}^{u} \xi(d\alpha, d\beta) = \frac{1}{2}\xi(R_{u,v}) = \frac{1}{2}\hat{W}_{u,v}.$$