25/7/16.

INTRO TO SPDE - SIVA.

LECTURE I :

[Walsh 1984] St. Flour Summer School.

Eg. 1 : $[0,1]$ - a rod of unit length. $u(0,x)$ - temp. at $x \in [0,1]$ at $t=0$.

No external heating / cooling

$$\frac{\partial u(t,x)}{\partial t} = c_1 \frac{\partial^2 u}{\partial x^2}$$

Dirichlet Boundary condn : $u : [0,\infty) \times [0,1] \to \mathbb{R}_+$.

(*)
$$\frac{\partial u}{\partial t} = c_1 \Delta u \qquad t > 0 \quad x \in [0,1]$$

$$u(0,x) = \phi(x). \quad u(t,1) = u(t,0) = 0. \quad t \geq 0.$$
$$x \in (0,1)$$

has a ! soln.

~~Eg. 2~~: External source of heating / cooling :

(*) - transforms to $\frac{\partial u}{\partial t} = c\,\Delta u + f(t,x)$ ↳ density of forcing
+ BC
↓
Additive force.

Dependence on u for external ~~force~~ (Multiplicative transform)

(*)
$$\frac{\partial u}{\partial t} = \Delta u + \sigma(u)\, F(t,x)$$

An Approach to SPDE : (Stochastic heat equation]

$F(t,x)$ - two parameter stochastic process / Random field / Noise / Gen. Random field.

Plan: Davar's book Ch. 1-4.

Focus : F = S-T white-noise denoted by $\xi_t(x) = \xi(t,x)$.

Intuition & Examples (today) + Rigorous defn. (~~in~~ future).

- white-noise $\xi(t,x)$ - mean 0 Gaussian process on $[0,\infty) \times [0,1]$

$$\mathrm{Cov}(\xi(t,x), \xi(s,y)) = \delta_0(t-s)\, \delta_0(x-y)$$

$\xi(t,x)$ & $\xi(s,y)$ are indep ~~...~~ $N(0,1)$ ~~iff~~ $t \neq s$ or $x \neq y$.

Example 2 : $\sigma(u) = \lambda u$ $\quad \frac{\partial u}{\partial t} = c_1 \Delta u + \lambda u \, \xi(t, u)$.

Case 1 : $\lambda = 0$, $c_1 = 1/2$ $\quad \varphi(x) = \sin(\pi x)$ + Dirichlet Bd. condns.

$$u(t,x) = e^{-\pi t/2} \sin(\pi x)$$

$\lambda = 0.1$, $\quad \lambda = 2$, $\quad \lambda = 5$. .

Nearly same. $\quad$ Appearance of spikes / Peaks $\rightarrow$ "Intermittency". $\quad$ — More pronounced.

SPDE $\subsetneq$ PDE

$u(t,x)$ - two parameter stochastic Process $\quad$ - Random coeff in PDE.

Aside : (other examples in Probability that lead to SPDE).

PARABOLIC ANDERSON MODEL :

$$\frac{\partial u}{\partial t} = \Delta u + \lambda u \, \xi(t, x) \quad \rightarrow \text{Random field.}$$

- applications. Random walk or BM in trap models.

Example 3 : (Measure-valued diffusion).

0-dim : $\{ Y_i^k : i \in \mathbb{N}, k \geq 1 \}$ - i.i.d mean 1, Variance γ

$$Z_n = \sum_{i=1}^{Z_{n-1}} Y_i^n \quad ; \quad Z_0 = 1, \; n \geq 1.$$

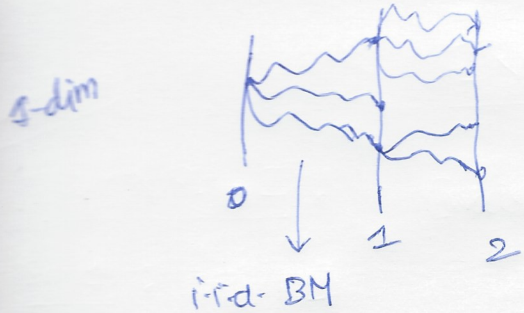
Critical G-W process. $\quad P(Z_k > 0) = \Omega\left(\frac{1}{k}\right)$

$$P\left(\frac{Z_k}{k} \in \cdot \;\middle|\; Z_k > 0 \right) \xrightarrow{d} \text{Exp. random variable.}$$

Feller's Branching diffusion. $\quad Z_0^n = n$. $\quad \{Z_k^n\}_{k \geq 1}$ - CGW

$$X_t^n = \frac{Z^n_{[nt]}}{n} \qquad X_t^n \xrightarrow{w} X_t \text{ in } D(\cdot) \text{ space.}$$

$$dX_t = \sqrt{\gamma X_t} \, dB_t, \quad X_0 = 1.$$



Branching Brownian motion.

$X_0^n = 1$. $X_t^n(A) = \dfrac{\#\text{ particles in } A \text{ at time } nt}{n}$

$$X^n_\cdot \in D([0,\infty) \to M_F(\mathbb{R}))$$

Dawson-Watanabe, . . . , Dynkin : $X_t^n \xrightarrow{w} X_t$ - Super BM.

d=1 $X_t(dx) = u(t,x)\,dx$. density. $\dfrac{\partial u}{\partial t} = \Delta u + \cancel{\tfrac{1}{2}}\sqrt{\gamma u}\,\xi(t,x)$. white noise

d≥2 Singular Measure.

General Branching Mechanism.

$$\frac{\partial u}{\partial t} = \Delta u + \sqrt{\gamma u f(u)}\,\xi(t,x).$$

Issues:

- Tightness, "Identifying limit", Uniqueness.

$\boxed{\text{Example 4:}}$ (Mueller-Tribe '97).

Long-range contact process & Voter Model.

$\frac{1}{n^2}\mathbb{Z}$ $x\sim y$ if $|x-y| = \frac{1}{\sqrt{n}}$ $x\in \frac{1}{n^2}\mathbb{Z}$ has $O(n^{3/2})$ neighbours.

$Z^{(n)}(t,x) \in \{0,1\}$ 0 - vacant 1 - occupied.

occupied site 1 $\xrightarrow{\text{at rate } n}$ vacant 0.

occupied site gives birth at rate $n+\theta_c$

Picks a site at random if occupied → cancel the birth
vacant → occupied.

$$u^n(t,x) = \frac{1}{n^{3/2}}\sum_{x\sim y} Z^n(t,y) \xrightarrow{w} u(t,x)$$

$$\frac{\partial u}{\partial t} = \frac{1}{6}\Delta u + \theta u - u^2 + \sqrt{2u}\,\xi(t,x).$$

VOTER MODEL : $\in\{0,1\}$. $\dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u + \theta u - \theta u^2 + \sqrt{u(1-u)}\,\xi(t,x)$