Due : Thursday, March 15th, 2002

1. (*Problem 9.22 in Karatzas and Shreve*) Show that for every $x > 0$, we have

$$\frac{x}{1+x^2}e^{-\frac{x^2}{2}} \leq \int_x^\infty e^{-\frac{u^2}{2}}du \leq \frac{1}{x}e^{-\frac{x^2}{2}}$$

2. Let $n \in \mathbb{N}$, $0 < \theta < 1$ and $\epsilon > \frac{(1+\theta)}{(1-\theta)} - 1$. Let Z be a N(0,1) random variable. Show that

$$2^n \sum_{k=1}^{[2n\theta]} P\left(\mid Z \mid \geq (1+\epsilon)\sqrt{\log\frac{4^n}{k^2}}\right) \leq \frac{c}{\sqrt{n}}2^{-\rho n},$$

where c is a constant and $\rho = (1-\theta)(1+\epsilon)^2 - (1+\theta)$.

3. (*Problem 9.26 in karatzas and Shreve*) Let θ, ϵ be as above. Consider the set $D = \cup_{n=1}^\infty D_n$ of dyadic rationals in $[0,1]$, with $D_n = \{k2^{-n} : k = 0, 1, \ldots, 2^n\}$. For every $\omega \in \Omega_\theta$ and every $n \geq N_\theta(\omega)$, the inequality

$$\mid B_t - B_s \mid \leq (1+\epsilon)\left[2\sum_{j=n+1}^\infty g(2^{-j}) + g(t-s)\right]$$

is valid for every pair (s,t) of dyadic rationals satisfying $0 < t - s < 2^{-n(1-\theta)}$

4. A function $f : [0,\infty) \to [0,\infty)$ is said to be Hölder-continuous with exponent γ if for all t, s

$$\mid f(t) - f(s) \mid \leq c \mid t - s \mid^\gamma,$$

for some constant c. Show that for almost every $\omega \in \Omega$, the Brownian path $B_\cdot(\omega)$ is nowhere Hölder-continuous with exponent $\gamma > \frac{1}{2}$