Due : Thursday, March 14th, 2002

1. Let x_n, x be real numbers. Let $\mathbb{P}_n$, $\mathbb{P}$ be measures on a probability space $(\Omega, \mathcal{F}$. Show that
   (a) $x_n \to x$ if and only if for every subsequence of x_n there exists a further subsequence that converges to x.
   (b) $\mathbb{P}_n \Rightarrow \mathbb{P}$ if and only if for every subsequence of $\mathbb{P}_n$ there exists a further subsequence that converges to $\mathbb{P}$.

2. Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a probability space. Let M be a fixed positive constant, f^n denote the nth derivative.
$$\mathcal{W} = \{f : \mathbb{R} \to \mathbb{R} \quad | \quad f^n \text{ exists and is continuous}, \quad \| f \|_\infty < M, \quad \| f^n \|_\infty < M\}.$$
   Show that $\mathcal{W}$ is a convergence determining class.

3. Let X_n, X be random variables on $\mathbb{R}^d$. Assume that every linear combination of the components of X_n converges in distribution to the corresponding linear combination of the components of X. Show that X_n converges weakly to X.

4. Let $\mathbb{P}_n$ (distribution of X_n) be a uniformly tight family of probability measures on $C([0,1])$, such that $\mathbb{P}_n\pi^{-1}_{t_1,\dots,t_k} \Rightarrow N(0,\Sigma)$, where $\Sigma = (t_i \wedge t_j)$, for all dyadic rationals $t_1, \dots, t_k \in [0,1], k = 1, 2, \dots$ Show that $X_n \Rightarrow B$, where B_t is a Brownian motion.

5. Let $\mathbb{P}$ be a probability measure on $(C([0,1]), \mathcal{C})$, under which the stochastic process $B(t)$ satisfies:
$$\mathbb{P}(B(0) = a) = 1 \tag{1}$$
$$\mathbb{P}(\{B(t) \leq \alpha\}) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\alpha} e^{-\frac{u^2}{2t}} du \tag{2}$$
$$\text{For } \ 0 \leq t_0 \leq t_1 \leq \cdots \leq t_k = 1 \ \ B(t_1) - B(t_0), B(t_2) - B(t_1), B(t_3) - B(t_2),$$
$$B(t_4) - B(t_3), \dots B(t_k) - B(t_{k-1}) \ \text{ are independent.} \tag{3}$$
   Let $\xi_1, \xi_2, \xi_3, \dots$.. be independent and normally distributed with mean 0 and variance 1 on $(\Omega, \mathcal{F}, P)$. Let $S_n = \sum_{i=1}^n \xi_i$. Let X_n be defined:
$$X_n(t) = \frac{1}{\sqrt{n}} S_{[nt]} + (nt - [nt]) \frac{1}{\sqrt{n}} \xi_{[nt]+1}$$
   (a) Show that X_n is a random variable on $(C([0,1])$.
   (b) Show that the finite dimensional distributions of X_n converge to the finite dimensional distributions of P.
   (c) Show that X_n is tight.
   (d) Conclude that X_n converges weakly to P and hence the weiner measure P exists.

6. Let $B(t)$ be as defined above. Let $\xi_1, \xi_2, \xi_3, \dots$.. be independent and distributed with Bernoulli$(\frac{1}{2})$ on $(\Omega, \mathcal{F}, P)$. Let $S_n = \sum_{i=1}^n \xi_i$. Let X_n be defined:
$$X_n(t) = \frac{1}{\sqrt{n}} S_{[nt]} + (nt - [nt]) \frac{1}{\sqrt{n}} \xi_{[nt]+1}$$
   (a) Using Donsker's theorem conclude that $\sup_t X_n(t) \Rightarrow B(t)$.
   (b) Show that $P\{\sup_t B(t) \leq \alpha\} = \frac{2}{\sqrt{2\pi}} \int_0^{\alpha} e^{-\frac{u^2}{2}} du, \alpha \geq 0$