Due : Thursday, January 31st, 2002

1. (*Time Inversion*) Let B_t be a standard Brownian motion in $\mathbb{R}$ (i.e $B_0 = 0$). Assume that $\lim_{t\to\infty} \frac{B_t}{t} = 0$ a.s.. Show that
$$\tilde{B}_t = \begin{cases} tB_{\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$
is also a standard Brownian motion.

2. (*Time Reversal*))Suppose B_t is a standard Brownian motion in $\mathbb{R}$. Let $T > 0$ be fixed. Then
$$\hat{B}_t = B_T - B_{T-t} : 0 \le t \le T$$
is a standard Brownian motion for $0 \le t \le T$.

3. (*Rotational Invariance*)Suppose B_t is a standard Brownian motion in $\mathbb{R}^d$. Let $A_{d\times d}$ be an orthogonal matrix. Show that $(\mathbb{P}^{Ax}, AX_t)$ is a Brownian motion, starting at Ax.

4. (*Loose ends in Existence proof*) We recall the notation discussed in class. Let $t \in [0,1], \phi_{00}(t) = 1, \psi_{00}(t) = t$. For $i = 1, 2, \ldots$ and $j = 1, 2, \ldots, 2^{i-1}$, Let
$$\phi_{ij}(t) = 2^{\frac{i-1}{2}} 1_{[\frac{(2j-2)}{2^i}, \frac{(2j-1)}{2^i}]}(t) - 2^{\frac{i-1}{2}} 1_{[\frac{(2j-1)}{2^i}, \frac{2j}{2^i}]}(t),$$
$\psi_{ij}(t) = \int_0^t \phi_{ij}(s)ds$, Y_{00}, Y_{ij} be an independent collection of Normal(0,1) random variables, and $V_i(t) = \sum_{j=1}^{2^{i-1}} Y_{ij}\psi_{ij}(t)$.
   (a) Assume that
   $$\sum_{i=1}^{\infty} P(V_i(t) > \frac{1}{i^2} \quad \text{for some} \quad t \in [0,1]) < \infty.$$
   Show that $B_t = \sum_{i=0}^{\infty} V_i(t)$ is well defined almost surely and the convergence is uniform in t.
   (b) Fix $t > 0$. Show that B_t is a Normal random variable.
   (c) Show that B_t is continuous a.s.

5. Let (S, ρ) be a metric space.
   (a) Define what is meant by saying: "S is a complete separable metric space".
   (b) Show that $\mathbb{R}^d$ is a separable metric space.
   (c) Let $S = C([0,1])$ be the space of real valued continuous functions. For any f, g define
   $$\rho(f,g) = \sup_{t\in[0,1]} \mid f(t) - g(t) \mid .$$
   Show that (S, ρ) is a complete separable metric space.