Due : Thursday, January 24th, 2002

1. State and prove the both parts of the Borel Cantelli Lemma.

2. Let X and Y be two random variables on a probability space $\{\Omega, \mathcal{F}, P\}$.
   (a) State precisely what is meant by the statement " X and Y are independent".
   (b) Suppose X and Y are as above. Construct a probability space and random variables X' and Y' such that X' and Y' are independent and X' has the same distribution as X and Y' the same distribution as Y.
   (c) Show that
   $$E(e^{iu(X+Y)}) = E(e^{iuX})E(e^{iuY})$$
   if and only if X and Y are independent.
   (d) Suppose X and Y are two Gaussian random variables and Variance$(X+Y)$ = Variance(X) + Variance(Y), then X and Y are independent.

3. If B_t is an $\mathcal{F}_t$ adapted Brownian motion starting at 0. Show that $\{-B_t : t \geq 0\}$ is also an $\mathcal{F}_t$ adapted Brownian motion starting at 0.

4. Show that a continuous $\mathcal{F}_t$ adapted stochastic process $\{B_t : t \geq 0\}$ is a Brownian motion starting at zero if and only if, for $n = 1, 2, \ldots$ and $0 \leq t_1 < t_2 < \ldots < t_n < \infty$, the n-dimensional random variables $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ has $N((0,\ldots,0), \Sigma_{n\times n})$ distribution where $\Sigma_{n\times n} = (t_i \wedge t_j)_{1\leq i,j\leq n}$.

5. Let a be a real number. Define
   $$Z_t(\omega) = \exp\{aB_t(\omega) - \frac{a^2 t}{2}\},$$
   where $t \geq 0, \omega \in C([0,\infty) : \mathbb{R})$ and B_t be a Brownian motion starting at 0. Show that Z_t is a martingale on $C([0,\infty) : \mathbb{R})$.