1. Consider the graph Γ obtained by identifying the origins of $\mathbb{Z}_+$ and $\mathbb{Z}^3$.
   (a) Show that Γ is transient.
   (b) Let $f_n : \Gamma \to \mathbb{R}$ be given by
   $$f_n(x) = \begin{cases} 1 - \frac{k}{n} & \text{if } x = k \text{ for } k \le n \\ 0 & \text{otherwise} \end{cases}$$
   Find f such that $f_n \to f$ in H^2. Conclude that H^2, H_0^2 and L^2 are all distinct spaces.
   (c) Let $h : \Gamma \to \mathbb{R}$ be given by
   $$h(x) = \begin{cases} \mathbb{P}^x(T_0 < \infty) & \text{if } x \in \mathbb{Z}^3 \\ 1 + 6p_0 n & \text{if } x = n \in \mathbb{Z}_+, \end{cases}$$
   where $p_o = \mathbb{P}^0($ R.W on $\mathbb{Z}^3$ does not return to 0). Show that h is non-constant positive harmonic function on Γ.
   (d) Show that Γ satisfies the Liouville property.
2. Consider the graph $\mathbb{Z}$ with natural weights. Calculate $\mathrm{R}_{\text{eff}}(\{0\}, [-n, n]^c)$. Use this to conclude that $\mathbb{Z}$ is recurrent.
3. Let $(\Gamma = (V, E), \mu)$ be a locally finite, connected, infinite vertex, weighted graph. Let $B_0, B_1 \subset V$. Let the effective conductance between them be given by
   $$\mathrm{C}_{\text{eff}}(B_0, B_1) = \inf\{\mathcal{E}(f, f) : f \mid_{B_1} = 1, f \mid_{B_0} = 0\}.$$
   (a) Let $B_0 = \{x\}$ and $B_1 = \{y\}$ where $x \sim y$ in V. Find $\mathrm{C}_{\text{eff}}(B_0, B_1)$.
   (b) Let $B_0 = \{x_1, x_2, x_3, \ldots x_n\}$ and $B_1 = \{y_1, y_2, y_3, \ldots y_n\}$ with only $x_i \sim y_i$ in V for $i = 1, 2, \ldots, n$. Find $\mathrm{C}_{\text{eff}}(B_0, B_1)$.
4. Let V be a countable graph, $A \subset V$, and $x_0 \in A$. Let $\mid A \mid < \infty$. Construct a unit flow from A^c to x_0. That is a flow $I : V \times V \to \mathbb{R}$ such that
   (a) $\mathrm{Div} I(x) = 0$ for all $x \in V \setminus (A^c \cup \{x_0\})$
   (b) $\mathrm{Div} I(x_0) = -1$
   (c) $\sum_{x \in A^c} \mathrm{Div} I(x) = 1$

   Hint: Consider a flow derived from a suitable Green function.
5. Let I be a flow on the weighted graph $(\Gamma = (V, E), \mu)$. Then show that $\| I \|_2^2 \le 2E(I, I)$.
6. Let $V = \{0, 1\} \times \mathbb{Z}_+$ and let $\Gamma = (V, E)$ be the (ladder) subgraph of Z^3 induced by V. Let $B_j = \{j\} \times \mathbb{Z}_+$ for $j = 0, 1$. If I^n be the flow such that $I^n_{(0,k),(1,k)} = \frac{1}{n}$ for $k = 1, 2, \ldots n$.
   (a) Find $E[I_n, I_n]$
   (b) Find $I \in \bar{\mathcal{I}}_o(B_0, B_1)$ such that $I_n \to I$.
   (c) Deduce that I is the minimiser of $\inf\{E[I, I] : I \in \bar{\mathcal{I}}_o(B_0, B_1)\}$
   (d) Conclude that we cannot replace $\bar{\mathcal{I}}_o(B_0, B_1)$ by $\mathcal{I}_o(B_0, B_1)$ in the above minimising problem.