HW1

G - be finitely generated. Λ_1, Λ_2 be two sets of generators for G.

Γ_1, Γ_2 be the associated Cayley graph

Then Γ_1 & Γ_2 are roughly isometric

$[(G, d_1), (G, d_2) \cdots]$

Proof: $\phi : G \to G$ $\phi(x) = x$

(1 b) immediate for all $C_2 \geq 0$

claim 1: $\exists\, C_{21} \geq 1$ such that $d_2(x,y) \leq C_{21}\, d_1(x,y)$ $\forall\, x, y \in G$

Claim 2: $\exists\, C_{12} \geq 1$ such that $d_1(x,y)^{-1} \leq C_{12}\, d_2(x,y)$ $\forall x, y \in G$

If claim 1 & claim 2 hold then take $C_1 = C_{12} \vee C_{21}$ & $C_2 = 0$ for (1.5)

Proof of Claim 1

Λ - generating set $\quad \Lambda^* = \{ g, g^{-1}$ such that $g \wedge g^{-1} \in \Lambda \}$

$\forall k \geq 1, \quad \Lambda_2(k) = \left\{ \prod_{j=1}^{m} x_j \;\middle|\; m \leq k \quad x_j \in \Lambda_2^* \right\}$

$x, y \in G \quad d_2(x,y) \leq k \iff \exists m \quad 1 \leq m \leq k \quad x_0 = x \to x x_1, x x_1 x_2 \dots = y = (\prod_{i=1}^{m} x_i) x$ with $x_i \in \Lambda_2^*$

$x \neq y$

$\iff x^{-1} y = \prod_{i=1}^{m} x_i \in \Lambda_2(k)$

for $1 \leq m \leq k \quad x_i \in \Lambda_2^*$

(ii) $z \in \Lambda_2(k_1)$ $w \in \Lambda_2(k_2)$

$$\Longrightarrow zw \in \Lambda_2(k_1+k_2)$$

From (i) & ii)

$$g \in \Lambda_1^* \Longrightarrow \exists 1 \leq k = k(g) \text{ such that}$$
$$g \in \Lambda_2(k)$$

$C_{21} = \max\{k(g) \mid g \in \Lambda_1^*\}$ as $|\Lambda_1^*| < \infty$

$C_{21} \geq 1$ &

$x, y \in G$ with $d_1(x,y) = m$

(as in proof of (i)) $\Leftrightarrow$ $x^{-1}y = \prod_{j=1}^{m} x_j \quad x_j \in \Lambda_1^*$

($x_j \in \Lambda_2(C_{21})$ & (ii)) $\Leftrightarrow$ $x^{-1}y \in \Lambda_2(m C_{21})$

(i) $\Leftrightarrow$ $d_2(x,y) \leq m C_2$

So $x, y \in G \Rightarrow d_2(x,y) \leq C_{21}\, d_1(x,y)$ $\square$

Proof of claim 2. - Interchange rôle of d_1 & d_2 in proof of claim 1 $\square$

claim 2.

If claim 1

Proof of Cla

$\forall k \geq 1$

(i) $x, y \in G$

Hewitt- Savage 0-1 law

let $\pi : \mathbb{N} \to \mathbb{N}$ be a finite permutation

π - 1-1 and onto

$\{ i \in \mathbb{N} \mid \pi(i) \neq i \}$ is finite

$(\Rightarrow \exists\, m = m(\pi)$ such that $\pi(i) = i \quad \forall\, i \geq m)$

$\Omega = V^{\mathbb{N}}$ $\qquad \xi_n : \Omega \to V$ $\qquad \mathcal{F} = \sigma(\xi_n,\ n \geq 0)$

$\xi_n(\omega) = \omega_n$

$F \in \mathcal{F}$

Theorem

(Hewitt Savage 0-

$M_\pi : \Omega \to \Omega$

$M_\pi(\omega) = (\omega_{\pi(i)})_{i \in \mathbb{N}}$

$F \in \mathcal{F}$ is said to permutable (exchangeable) if $F = M_\pi^{-1}(F)$

$= \{\omega \in \Omega \mid M_\pi(\omega) \in F\}$

<u>Theorem</u>: $\mathcal{F}_E = \{F \in \mathcal{F} \mid F$ is exchangeable$\}$ the $\mathcal{F}_E$ is a σ-field, and is called the exchangeable σ-field

(Hewitt Savage 0-1). $\{\xi_n : n \geq 1\}$ is iid in $(\Omega, \mathcal{F}, P)$ then $\mathcal{F}_E$ is trivial under P

Corollary

Proof

$\xi_n \stackrel{d}{=}$

$\Rightarrow X_n \stackrel{d}{=}$

$(\xi_n =$

Corollary: X a simple random walk on $\mathbb{Z}^d$. Then $\mathcal{J}$ & $\mathcal{I}$ are trivial & consequently $\mathbb{Z}^d$ satisfies strong Liouville property

Proof: $S = \{ e_i \mid 1 \le i \le 2d \}$

$(1, 0, \dots, 0)$
$(-1, 0, \dots, 0)$
$(0, 1, \dots,)$

$\mathbb{P}^{x_0}(X_0 = x_0) = 1$

$\xi_n \overset{d}{=}$ i.i.d. Uniform (S)

$\Rightarrow X_n \overset{d}{=} x_0 + \sum_{i=1}^{n} \xi_i$ (Ex)

$(\xi_n = X_n - X_{n-1})$

$\mathcal{F}_n = \sigma(\xi_k : k \le n) = \sigma(X_k : k \le n)$

$\mathcal{G}_0 = \sigma(X_k ; k \ge 0) = \sigma(\xi_k ; k \ge 0)$

$\mathcal{I} = \{ F \in \mathcal{G}_0 \mid \Theta_n^{-1}(F) = F \quad \forall n \geq 1 \}$

$\mathcal{J} = \bigcap_{n \geq 0} \mathcal{G}_n$

$\boxed{\text{HS - 0-1 law} \Rightarrow \mathcal{J}_E \text{ is } \mathbb{P}^{x_0} \text{ trivial}}$ — (*)

let $\pi : \mathbb{N} \to \mathbb{N}$ be a finite permutation

$X_n^{\pi} = \sum_{i=1}^{n} \xi_{\pi(i)} + x_0$

$\exists\ m \equiv m(\pi)$ such that $\pi(i) = i \quad \forall\, i \geq m$

$$\Rightarrow \quad X_n^{\pi} = X_n \quad \forall\, n \geq m$$

So $\quad G \in \mathcal{G}_n \Rightarrow G \in \mathcal{F}_E \quad \forall\, n \geq m$

$$\Rightarrow \quad \mathcal{J} \subseteq \mathcal{F}_E$$

But (*) that $\mathcal{J}$ is trivial under $\mathbb{P}^{x_0}$

$\mathcal{I} \subseteq \mathcal{J}$ So $\mathcal{I}$ is trivial under $\mathbb{P}^{x_0}$ $\xRightarrow{\text{4 in HW3}}$ $\mathbb{Z}^d$ satisfies strong Liouville property