Recall.

$h : V \to V$

$$\Delta h(x) = (P - I)h(x) = \frac{1}{\mu_x} \sum_{y \sim x} \mu_{xy} (h(y) - h(x))$$

(Maximum Principle)

$A \subseteq V$ - connected $\quad \Delta h \geq 0$ in $A$

(a) If $\exists\, x \in A$ such that

$$h(x) = \max_{z \in \bar{A}} h(z) \implies h \text{ is constant on } \bar{A}.$$

(b) A is finite

Liouville Property

Property

It has the s

are constant

(b) $A$ is finite, then it attains maximum on $\bar{A}$ &

$$\max_{x \in \partial A} h(x) = \max_{x \in \bar{A}} h(x)$$

Liouville Property: $\underset{\substack{\parallel \\ (V, E)}}{(\Gamma, \mu)}$ be a weighted graph. It has Liouville property if all bounded harmonic $\underset{(\Delta h = 0)}{\text{function}}$ on $V$ are constant.

It has the Strong Liouville property if all positive harmonic function on $V$ are constant.

• $(P, \mu)$ is recurrent, then any non-negative superharmonic function is constant

(in particular satisfies strong Liouville property)

• [Foster's Criterion] $A \subseteq V$ be finite

$(P, V)$ is recurrent $\iff \exists\, h : V \to V$, non-negative superharmonic function on $V \setminus A$ & has the property $|\{x : h(x) < M\}| < \infty \quad \forall M > 0$

## Harnack Inequalities

**Proposition 1.43** $(P, \mu)$ has controlled weights, (H5) with constant $C_2$