

Recall:

$X: S \rightarrow \mathbb{R}$ is continuous r.v. $\Leftrightarrow$

$$P(a < X \leq b) = \int_a^b f(x) dx$$

f - probability density function $\left\{ \begin{array}{l} f \geq 0 \\ f \text{ - piecewise continuous} \\ \int_{-\infty}^{\infty} f(x) dx = 1 \end{array} \right.$

X is continuous $\Rightarrow P(X=a) = 0 \quad \forall a \in \mathbb{R}$

$X \sim \text{Uniform}(a, b)$

$-\infty \leq a < b < \infty$

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$X \sim \text{Exponential}(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P(X > t+s | X > s) = P(X > t)$$

(Memoryless Property)

$X \sim \text{Normal}(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

X - Gaussian random variable with parameter μ and σ^2 .

Theorem (De Moivre - Central Limit Theorem) Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of iid random variable $X_i \sim \text{Bernoulli}(p)$ $0 < p < 1$

$$S_n = \sum_{i=1}^n X_i \quad Y_n = \frac{S_n - np}{\sqrt{np(1-p)}}$$

$$\lim_{n \rightarrow \infty} P(a < Y_n \leq b) = \int_a^b \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \quad -\infty < a < b < \infty$$

Remarks:-

• Theorem holds without $X_i \sim \text{Bernoulli}(p)$ assumption.
 If $E[X_i] = \mu$ $\text{Var}[X_i] = \sigma^2$, $S_n = \sum_{i=1}^n X_i$ $Y_n = \frac{S_n - n\mu}{\sqrt{n}\sigma}$ } Central limit Theorem

$$\lim_{n \rightarrow \infty} P(a < Y_n \leq b) = \int_a^b \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

Proof (Sketch)

$$S_n \sim \text{Binomial}(n, p)$$

$$P(a < Y_n \leq b) = P\left(a < \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\right)$$

$$= P(a\sqrt{np(1-p)} + np < S_n \leq \sqrt{np(1-p)} b + np)$$

$$= \sum_{k \in A_n} P(S_n = k)$$

where

$$= \sum_{k \in A_n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$A_n = \left\{ k \in \mathbb{N} \mid a\sqrt{np(1-p)} + np < k \leq b\sqrt{np(1-p)} + np \right\}$$

$$= \sum_{k \in A_n} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{np(1-p)}} e^{-\frac{1}{2} \left(\frac{k-np}{\sqrt{np(1-p)}} \right)^2} - \sum_{k \in A_n} \frac{e^{-\frac{1}{2} \left(\frac{k-np}{\sqrt{np(1-p)}} \right)^2}}{\sqrt{2\pi} \sqrt{np(1-p)}} + \sum_{k \in A_n} n_{k,n} p^k (1-p)^{n-k}$$

$$= \sum_{k \in A_n} \frac{e^{-\frac{1}{2} \left(\frac{k-np}{\sqrt{np(1-p)}} \right)^2}}{\sqrt{2\pi} \sqrt{np(1-p)}} + \sum_{k \in A_n} \frac{e^{-\frac{1}{2} \left(\frac{k-np}{\sqrt{np(1-p)}} \right)^2}}{\sqrt{2\pi} np(1-p)} \left[\frac{n_{k,n} p^k (1-p)^{n-k} \cdot \sqrt{2\pi} np(1-p)}{e^{-\frac{1}{2} \left(\frac{k-np}{\sqrt{np(1-p)}} \right)^2}} - 1 \right]$$

Riemann Approximation
Analysis 2

$n \rightarrow \infty$

$$\int_a^b \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

$\lim_{n \rightarrow \infty} \frac{n_{k,n} p^k (1-p)^{n-k} \cdot \sqrt{2\pi} np(1-p)}{e^{-\frac{1}{2} \left(\frac{k-np}{\sqrt{np(1-p)}} \right)^2}} = 1$
 $\sum_{k \geq 0} \frac{e^{-\frac{1}{2} \left(\frac{k-np}{\sqrt{np(1-p)}} \right)^2}}{\sqrt{2\pi} np(1-p)} = 1 \quad \text{for } \forall \epsilon > 0 \exists N$
 $|b_{n,k} - 1| < \epsilon \quad \forall n > N$
 $k \in A_n$

Usage

Computing
Numerator
Stored in N

Usage

$$P(\alpha < S_n \leq \beta) = P\left(\frac{\alpha - np}{\sqrt{np(1-p)}} < \frac{S_n - np}{\sqrt{np(1-p)}} \leq \frac{\beta - np}{\sqrt{np(1-p)}}\right)$$

D. (CLT)

$$\int_{\frac{\alpha - np}{\sqrt{np(1-p)}}}^{\frac{\beta - np}{\sqrt{np(1-p)}}} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

Computing $\int_a^b \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$ is Hard

} Numerically done

Stored in Normal Tables

$$X \sim \text{Normal}(\mu, \sigma^2) \quad \sigma \neq 0$$

calculate $P(a < X \leq b) = \int_a^b \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} dx$?

let $Y = \frac{X-\mu}{\sigma}$

$$-a \leq a \leq b \leq a, P(a < Y \leq b) = P(a < \frac{X-\mu}{\sigma} \leq b)$$

$$= P(a\sigma + \mu < X \leq b\sigma + \mu)$$

$$\Rightarrow Y$$

$$X \sim \text{Normal}(\mu, \sigma^2)$$

$$\sigma > 0$$

$$\sigma \neq 0$$

p.d.f.

$$f(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} dx$$

calculate $P(a < X \leq b) = \int_a^b \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} dx$?

let $Y = \frac{X-\mu}{\sigma}$

$$-a \leq a \leq b \leq a, P(a < Y \leq b) = P(a < \frac{X-\mu}{\sigma} \leq b)$$

$$= P(a\sigma + \mu < X \leq b\sigma + \mu)$$

$$= \int_{a\sigma + \mu}^{b\sigma + \mu} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} dx$$

$$, \quad y = \frac{x-\mu}{\sigma} \quad dy = \frac{1}{\sigma} dx$$

$$= \int_a^b \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$$

$$x = a\sigma + \mu \iff y = a$$

$$x = b\sigma + \mu \iff y = b$$

$$\Rightarrow Y \text{ has p.d.f. } g(y) = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \quad -\infty < y < \infty$$

$$Y \sim \text{Normal}(0, 1)$$

$$\text{Ex. } Z \sim \text{Normal}(0, 1)$$

$$W = \alpha Z + \beta \quad \alpha, \beta \in \mathbb{R}$$

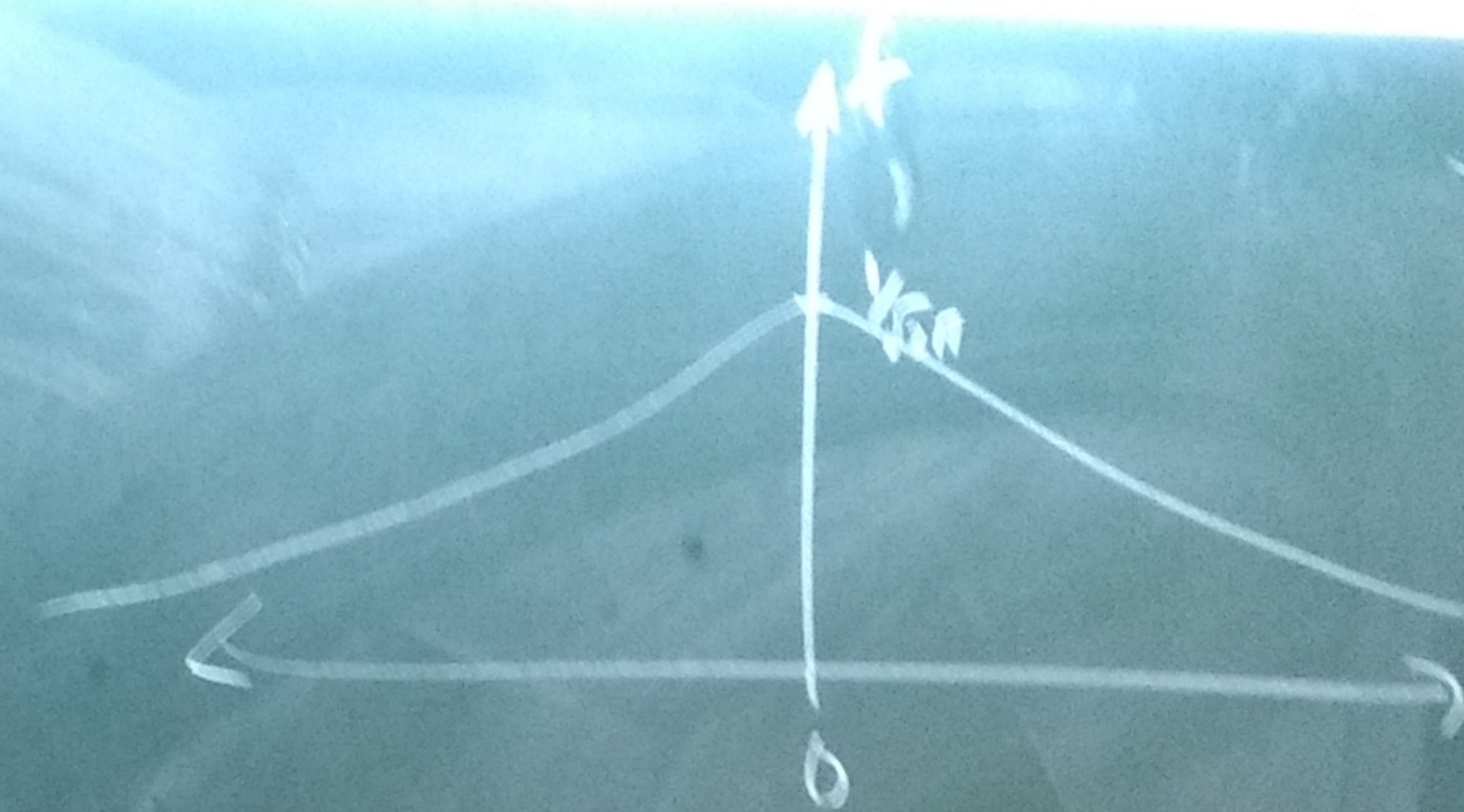
$$\Downarrow \\ W \sim \text{Normal}(\beta, \alpha^2)$$

$$Z \sim \text{Normal}(0,1)$$

p.d.f
of Z

$$f(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$$

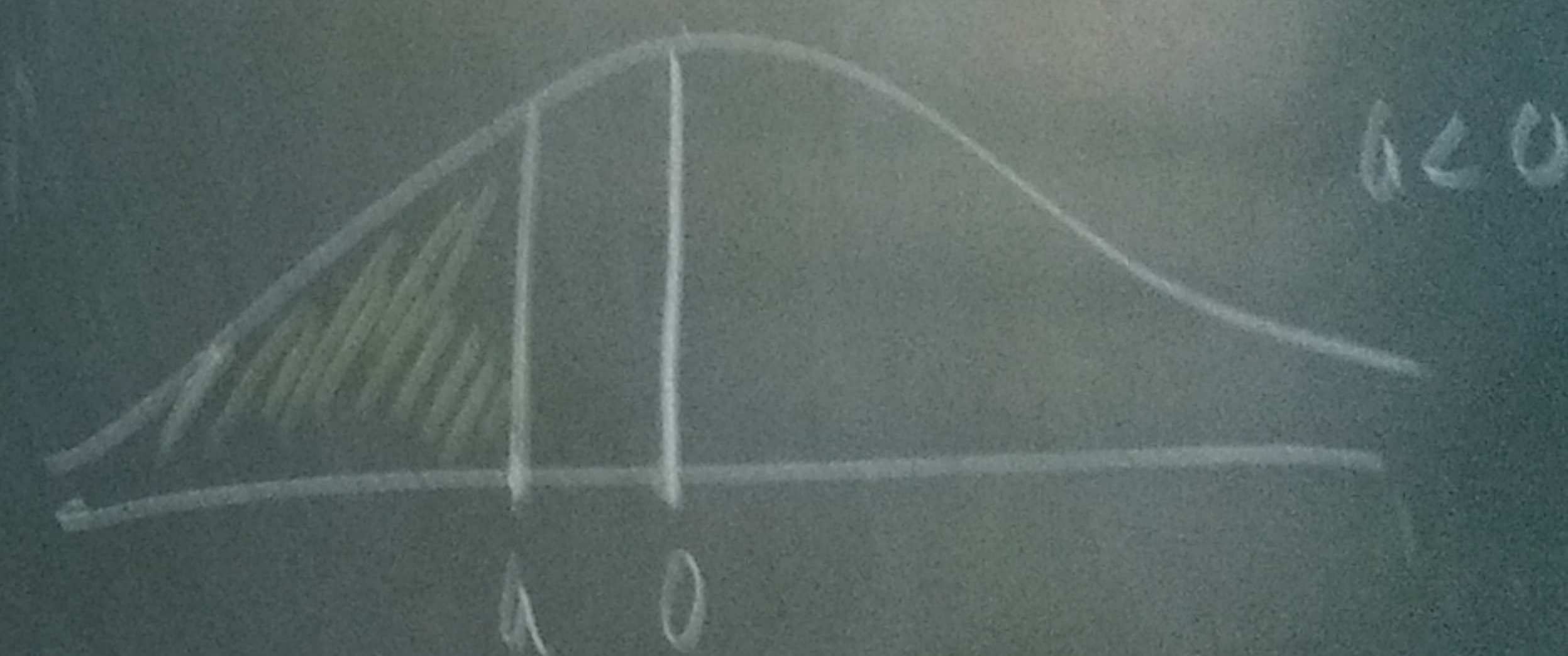
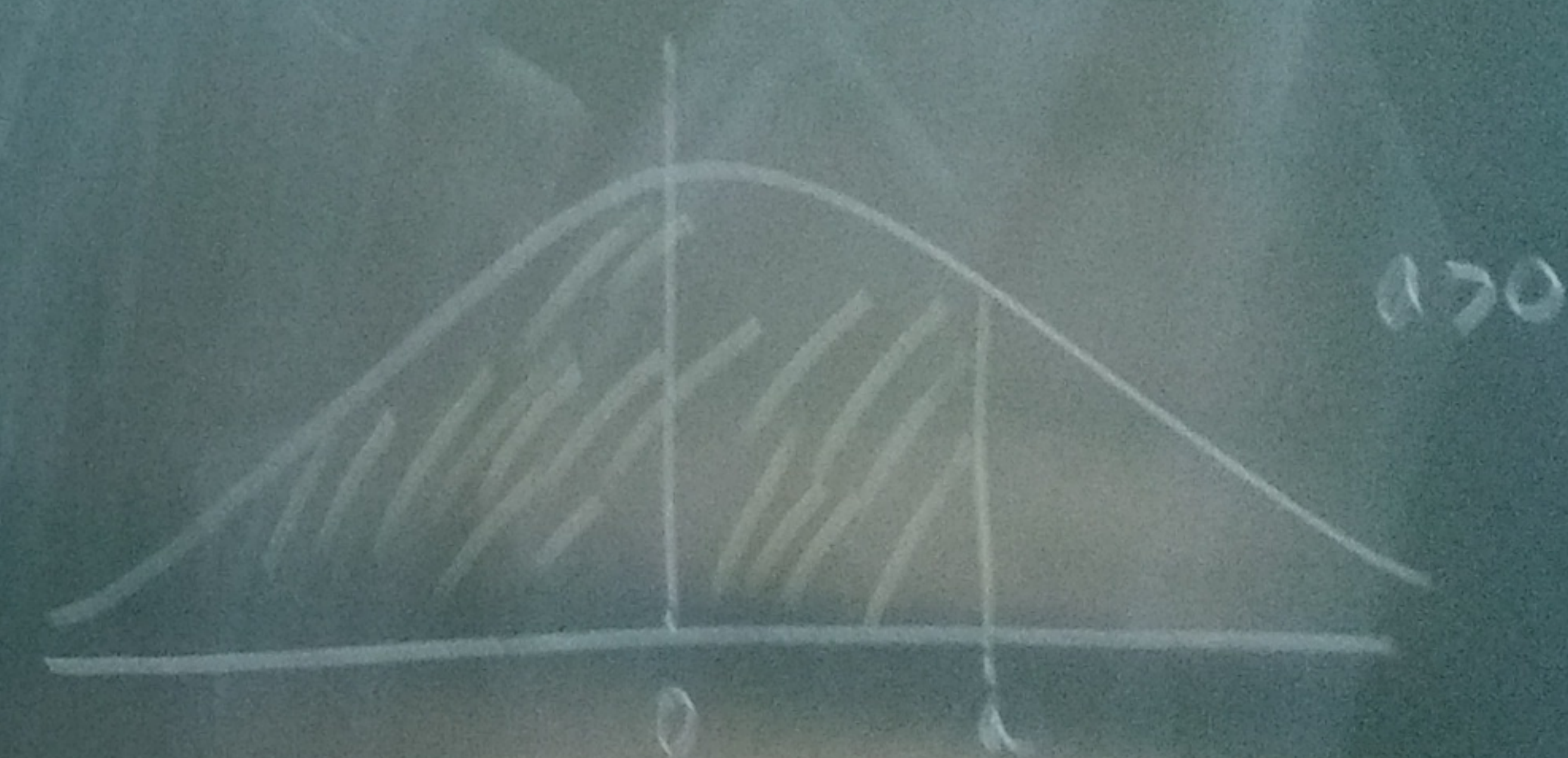
$$-\infty < z < \infty$$



Symmetry $f(z) = f(-z)$

$$\Phi(a) = P(Z \leq a) = \int_{-\infty}^a \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$$

Distribution
function of
 Z



$$1 = \int_{-\infty}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = \int_{-\infty}^0 \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz + \int_0^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = 2\Phi(0)$$

$\underbrace{u = -z} \Rightarrow \int_{-\infty}^0 \frac{e^{-u^2/2}}{\sqrt{2\pi}} du$

$\swarrow$
 f is a p.d.f