

Recall:

- Repeated Trials - { Experiment & A-event of interest
Success - if A occurs
Failure - otherwise

(Independent)

- trials conducted & measure the number of
Successes

Example: "Viewed as sampling from a large population"

Factory manager - "bolt"

Q. how many defective bolts are made in a day?

X 10 days, on each day you pick a random bolt,

check if is defective or not

- Success if ball is not defective
failure otherwise

$P(\# \text{ of failures in } 100 \text{ days})$

Bernoulli (p) Trials $0 \leq p \leq 1$

- Suppose we perform an experiment, many times (independent trials)

Sequence of trials is called Bernoulli(p) trial

Each trial $\begin{cases} \text{success} \\ \text{failure} \end{cases}$

$p = P(\{\text{success}\})$

one trial - write -

$S = \{\text{success}, \text{failure}\}$

$P(\{\text{success}\}) = p$

$P(\{\text{failure}\}) = 1 - p$

Example 2.1.1 ($\sim$ Example 1.4.3.)

Q: What is the probability that a six appears exactly once when we roll a die two times?

• View each roll as a Bernoulli(p) trial.

$$\text{i.e. } \begin{cases} \text{Success} = \{6 \text{ appears}\} & \text{failure} = \text{otherwise} \\ P(\{\text{Success}\}) = 1/6 \end{cases}$$

Bernoulli($1/6$)

$\sim$ Probability of exactly one success in 2 Bernoulli($1/6$) trials?

So far we would approach it in the following manner:-

$$S = \{ \{\text{Success, Success}\}, \{\text{Success, failure}\}, \{\text{failure, Success}\}, \{\text{failure, failure}\} \}$$

Recall

$$P(\{ \text{succes}, \text{succes} \}) = P(\{ \text{succes} \}) P(\{ \text{succes} \}) \\ = \frac{1}{36}$$

$$P(\{ \text{succes}, \text{failure} \}) = P(\{ \text{succes} \}) P(\{ \text{failure} \}) \\ = \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{36}$$

$$P(\{ \text{failure}, \text{succes} \}) = \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{36}$$

$$P(\{ \text{failure}, \text{failure} \}) = \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{36}$$

Ans

Que

$$\begin{aligned}
 P(\text{exactly one six appears}) &= P(\text{one success in two trials}) = P(\{\text{Success, failure}\} \cup \{\text{failure, success}\}) \\
 &= P(\{\text{success, failure}\}) + P(\{\text{failure, success}\}) \\
 &= \frac{5}{36} + \frac{5}{36} = \frac{10}{36}
 \end{aligned}$$

Another approach - $\tilde{S} = \left\{ \begin{pmatrix} 1,1 \\ \vdots \\ 6,1 \end{pmatrix}, \begin{pmatrix} 1,6 \\ \vdots \\ 6,6 \end{pmatrix} \right\}$, all 36 possibilities, equally likely

$$P(\text{exactly one six in two trials}) = P\left(\left\{ \begin{pmatrix} 1,6 \\ 6,1 \end{pmatrix}, \begin{pmatrix} 2,6 \\ 6,2 \end{pmatrix}, \begin{pmatrix} 3,6 \\ 6,3 \end{pmatrix}, \begin{pmatrix} 4,6 \\ 6,4 \end{pmatrix}, \begin{pmatrix} 5,6 \\ 6,5 \end{pmatrix} \right\}\right) = \frac{10}{36}$$

Question:- Can we do this in general? (without rewriting the structure of experiment)

Exa

(a)

(b)

(c)

(d)

Example 2.1.2

Given n - ^{independent} Bernoulli(p) trials, $n \in \mathbb{N}$.

- (a) $0 \leq k \leq n$ $\Rightarrow$ $\mathbb{P}$ (there are k success in n trials) ?
- (b) most likely number of success in trials ?
- (c) how many trials are required to obtain the first success ?
- (d) On an average how many successes will there be in n trials ?

① - $n=1$ ✓

$$P(0-\text{succ}) = 1-p$$

$$P(1-\text{succ}) = p$$

$n \geq 1$

$$W = \{w_1, w_2, \dots, w_n\}$$

where each w_i $1 \leq i \leq n$

represents the outcome in the i^{th} trial

i.e. $w_i \in \{\text{succ}, \text{failure}\}$

$$\cdot \underline{P(\{w_i\}) = ?}$$

independence

$$P(\{w_1, \dots, w_n\}) = P\left(\bigwedge_{i=1}^n \{w_i\}\right) = \prod_{i=1}^n P(\{w_i\})$$

Note

$$P(\{w_i\}) = \begin{cases} p & y \quad w_i = \text{success} \\ 1-p & y \quad w_i = \text{failure} \end{cases} \quad 1 \leq i \leq n.$$

$$P(\{w\}) = \prod_{i=1}^n P(\{w_i\}) = p^{\#\{i: w_i = \text{success}\}} \cdot (1-p)^{\#\{i: w_i = \text{failure}\}}$$

$B_k = \{k - \text{successes in } n\text{-trials}\}$

$$P(B_k) = \sum_{w \in B_k} P(\{w\}) = \sum_{w \in B_k} p^{\#\{i: w_i = \text{success}\}} (1-p)^{\#\{i: w_i = \text{failure}\}}$$

$\rightarrow p^k (1-p)^{n-k}$

$$w \in B_k \Rightarrow$$

$$\# \{i : w_i = \text{success}\} = k$$

$$\# \{i : w_i = \text{failure}\} = n - k$$

$$P(B_k) = |B_k| p^k (1-p)^{n-k}$$

$$= \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \leq k \leq n.$$

Choose k for success

$$a > 0, b > 0$$

$$a, b : a+b > 0$$

$$\left(\frac{a}{a+b} + b/a+b \right)^n = \sum_{k=0}^n \binom{n}{k} \frac{a^k}{(a+b)^n}$$

Fact:

$$1 = P(S) = P\left(\bigcup_{k=0}^n B_k\right) = \sum_{k=0}^n P(B_k) = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k}$$

If we were interested solely on number of
 successes then we could view our
 sample space as:

$$S = \{0, 1, 2, \dots, n\} \quad \text{ends as} \quad \mathcal{F} \equiv \text{all subsets of } S$$

$$P: \mathcal{F} \rightarrow [0, 1]$$

$$P(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\& \quad P(A) = \sum_{w \in A} P(w)$$

$$K \in S$$

(b)

Note: The distribution of probabilities on $\{0, 1, \dots, n\}$ is called Binomial(n, p)

$$n=1 \quad \text{Binomial}(n, p) \equiv \text{Bernoulli}(p)$$

$$S = \{0, 1\}$$

$$P(\{0\}) = 1-p \quad P(\{1\}) = p$$

(b) Mode of a Binomial(n, p)

$$\begin{array}{l} p=0 \quad \& \quad p=1 \quad \checkmark \\ m=0 \quad \quad m=n \end{array}$$

$$0 < p < 1$$

Find k such that

$$P(B_k) \text{ is maximised}$$

Given n : Find k : $\binom{n}{k} p^k (1-p)^{n-k}$ is maximized

$$\frac{P(B_{k+1})}{P(B_k)} = \frac{\binom{n}{k+1} p^{k+1} (1-p)^{n-(k+1)}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{n-k}{k+1} \frac{p}{1-p}$$

$$P(B_{k+1}) \geq P(B_k) \Leftrightarrow \frac{n-k}{k+1} \frac{p}{1-p} \geq 1 \Leftrightarrow (n-k)p \geq (k+1)(1-p)$$

$$\Leftrightarrow np - (1-p) \geq k \Leftrightarrow k \leq (n+1)p - 1$$

"most likely # of success"
Mode of Binomial $p \in k$

$$k-1 \leq (n+1)p - 1 < k$$

$$k = \lfloor (n+1)p \rfloor$$

(5) $\{ \omega \in \Omega : \text{the process } X \text{ is a success} \}$
 is a σ -algebra. (if success happens)
 (6) $\{ \omega \in \Omega : \text{the process } X \text{ is a failure} \}$
 is a σ -algebra. (if failure happens)

$$\begin{aligned}
 P(A) &= \sum_{i=1}^n P(A_i) \\
 P(A) &= \sum_{i=1}^n P(A_i) \\
 P(A) &= \sum_{i=1}^n P(A_i) \\
 P(A) &= \sum_{i=1}^n P(A_i)
 \end{aligned}$$

$$= (1-p)^{k-1} p, k \geq 1$$

Geometric (p)

$$0 < p < 1$$

$S = \mathbb{N}$, $\mathcal{F} \equiv \text{events} \equiv \text{all subsets of } S$

$$P: \mathcal{F} \rightarrow [0,1]$$

$$P(\{k\}) = (1-p)^{k-1} p$$

$$k \in \mathbb{N}$$

Exercise
 P defines
a probability

$$A \subseteq \mathbb{N}$$

$$P(A) = \sum_{k \in A} P(\{k\})$$

(d) - answer soon.

Example 2.1.3

Ted & Sania play a tennis match of 3 sets.

The winner is the person who wins the sets, out of 3.

Sania can win each set with probability $\frac{2}{3}$.

How likely is it that Sania will win the match?

Approach

View the above as Bernoulli ($\frac{2}{3}$) trials

$$\begin{aligned} P(\text{Sania wins the match}) &= P(\{WWW\} \cup \{LWW\} \cup \{WLW\}) \\ &= P(\{WWW\} \cup \{LWW\} \cup \{WLW\}) \end{aligned}$$

$$\begin{aligned}
 &= P(\{wwL\}) + P(\{wLw\}) + P(\{Lww\}) + P(\{LwL\}) \\
 &= \frac{4}{27} + \frac{8}{27} + \frac{4}{27} + \frac{4}{27} = \frac{20}{27}
 \end{aligned}$$

2.2 Poisson Approximation

Computation of Binomial(n, p) probabilities when
 n -large is challenging

Try to compute
 $P(X = k)$

$$\rightarrow \binom{n}{k} p^k (1-p)^{n-k}$$

Example 2.2.1

A small college has 1460 students. Assume that the birth "rate" is constant over a year & each year has 365 days.

What is the probability that 5 or more people are born on

August 15th?

— Think : each person has $\frac{1}{365}$ "probability" of being born on August 15th.

Binomial(1460, $\frac{1}{365}$) — Experiment

$P(\text{there are 5 or more successes in a Binomial}(1460, \frac{1}{365}) \text{ experiment})$?

Theorem 2.2.2

$$P(\bigcup_{k=5}^{1460} \{k \text{ success in } 1460 \text{ Bernoulli}(1/365) \text{ trials}\})$$

$$= \sum_{k=5}^{1460} {}^{1460}C_k \left(\frac{1}{365}\right)^k \left(1 - \frac{1}{365}\right)^{1460-k}$$

$$= 1 - \sum_{k=0}^4 {}^{1460}C_k \left(\frac{1}{365}\right)^k \left(1 - \frac{1}{365}\right)^{1460-k}$$

Theorem 2.2.2: let $\lambda > 0$, $k \geq 1$, $\lambda > 0$, $p = \frac{\lambda}{n}$, $n \geq \lceil \lambda \rceil + 1$

$$P(k \text{ success in } n \text{ Bernoulli}(p) \text{ trials}) = {}^nC_k \left(\frac{\lambda}{n}\right)^k (1 - \frac{\lambda}{n})^{n-k} \approx \frac{\lambda^k}{k!} e^{-\lambda}$$