Get to Bivariate Normal

Conditional Expectation & Variance

Definition 6.3.3: Let (X,Y) be a continuous random variable with joint p.d.f $f(\cdot,\cdot)$

Let $f_X(\cdot)$ be the p.d.f of X.

$x \in \mathbb{R}$, $f_X(\cdot)$ is continuous at x & $f_X(x) > 0$

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} \quad y \in \mathbb{R}, \quad \boxed{Y|X=x}$$

hat to Bivariate Normal

random

>0

$Y|X=x$

$$E[Y|X=x] := \int_{-\infty}^{\infty} y\, f_{Y|X=x}(y)\, dy = \int_{-\infty}^{\infty} y\, \frac{f(x,y)}{f_X(x)}\, dy$$

Conditional expectation of Y given that $X=x$.

$$\mathrm{Var}[Y|X=x] = E\left[(Y - E[Y|X=x])^2 \,\middle|\, X=x \right]$$

$$= \int_{-\infty}^{\infty} (y - \underbrace{E[Y|X=x]}_{\text{constant in 'y'}})^2 \frac{f(x,y)}{f_X(x)}\, dy$$

Conditional variance of Y given $X=x$

$$E[h(Y)|X=x] := \int_{-\infty}^{\infty} h(y) f_{Y|X=x}(y)\, dy$$

Recall

$\mathrm{Var}[Z] = E[Z^2] - E[Z]^2$

- Discrete case :-

various facts about conditional expectation & conditional variance

$\mathrm{Var}[Y] = E[\mathrm{Var}[Y|X]] + \mathrm{Var}[E[Y|X]]$

Facts:

① $$\mathrm{Var}[Y|X=x] = \frac{1}{f_X(x)}\left[\int_{-\infty}^{\infty}\left(y^2 - 2y\,E[Y|X=x] + \left(E[Y|X=x]\right)^2\right) f(x,y)\,dy\right.$$

$$= \int_{-\infty}^{\infty} y^2 \frac{f(x,y)}{f_X(x)}\,dy - 2E[Y|X=x]\int_{-\infty}^{\infty} y\,\frac{f(x,y)}{f_X(x)}\,dy + \frac{\left(E[Y|X=x]\right)^2}{f_X(x)}\int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= E[Y^2|X=x] - 2\left(E[Y|X=x]\right)^2 + \left(E[Y|X=x]\right)^2$$

$$= E[Y^2|X=x] - \left(E[Y|X=x]\right)^2 \qquad \square$$

$E[Y|X]$

(2) $x \in \mathbb{R}$

$g: \mathbb{R} \to \mathbb{R}$

$$g(x) = \begin{cases} E[Y \mid X = x] & f_X(\cdot) \text{ is continuous at } x \text{ \& } f_X(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

<u>Black Box</u> :- $f(\cdot, y)$ is p.c $\Rightarrow$ g is p.c.

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{y f(x,y)\, dy}{\cancel{f_X(x)}} \right] \cancel{f_X(x)}\, dx$$

$\boxed{\text{Issue to deal with } f_X(x) = 0}$

Reorder integrals $= \int_{-\infty}^{\infty} y \left[\int_{-\infty}^{\infty} f(x,y)\, dx \right] dy = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = E[Y]$

Notation :- $g(X) =$ " $E[Y|X]$ "

Short : $E[E[Y|X]] = E[Y]$

③ $x \in \mathbb{R}$ $\quad g : \mathbb{R} \to \mathbb{R}$

$$g(x) = \begin{cases} E[Y|X=x] & f_X(\cdot) \text{ is continuous at } x \text{ \& } f_X(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$h : \mathbb{R} \to \mathbb{R}$

$$h(x) = \begin{cases} Var[Y|X=x] & f_X \text{ con… \&} \\ 0 & \text{othe…} \end{cases}$$

$$Var[g(X)] = E[g^2(X)] - E[g(X)]^2 \underset{\uparrow \, ②}{=} E[g^2(X)] - (E[Y])^2$$

$$E[h(X)] \underset{\uparrow \, ①, ②}{\overset{(Ex)}{=}} E[Y^2] - E[g^2(X)]$$

observe:

$h(x) = E[Y^2 | X = x] - g^2(x)$

$$Var[g(X)] + E[h(X)] = E[Y^2] - E[Y]^2 = Var[Y]$$

Notation:- $Var[Y] = Var[E[Y|X]] + E[Var[Y|X]]$

Example 6.1.5 (Contd.)

$$f(x,y) = \frac{\sqrt{3}}{4\pi} e^{-\frac{(x^2 - xy + y^2)}{2}} \qquad x, y \in \mathbb{R}$$

$$\Rightarrow \quad X \sim \text{Normal}(0, 4/3)$$
$$Y \sim \text{Normal}(0, 4/3)$$

$x \in \mathbb{R}$ $\quad Y \mid X = x$; $\quad f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} \qquad y \in \mathbb{R}$

$$= \frac{\frac{\sqrt{3}}{4\pi} e^{-\frac{(x^2 - xy + y^2)}{2}}}{\frac{1}{\sqrt{2\pi}} \frac{\sqrt{3}}{2} e^{-\frac{3}{8}x^2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y - \frac{x}{2}\right)^2}$$

$$Y \mid X = x \sim \text{Normal}\left(\frac{x}{2}, 1\right)$$

$$E[Y \mid X = x] = \frac{x}{2} \quad \& \quad \text{Var}[Y \mid X = x] = 1$$

$$\text{Var}[Y] = \text{Var}[E[Y \mid X]] + E[\text{Var}[Y \mid X]]$$

$$= \text{Var}\left[\frac{X}{2}\right] + E[1]$$

$$= \frac{1}{4}\text{Var}[X] + 1$$

$$= \frac{1}{3} + 1 = 4/3$$