Due: Tuesday, April 13, 2010

1. Let $t \in \mathbb{R}$, $i = \sqrt{-1}$, and $n \in \{0\} \cup \mathbb{N}$. Then show that

$$|e^{it} - \sum_{k=0}^{n} \frac{it^k}{k!}| \leq \min(\frac{|t|^{n+1}}{n+1!}, 2\frac{|t|^n}{n!})$$

2. Let X_n be a sequence of independent and identically $N(\mu, 1)$ distributed random variables. Let $S_n = \sum_{i=1}^{n} X_i$. Define

$$Y_n = \begin{cases} \frac{S_n}{n} & \text{if } |S_n| > n^{\frac{3}{4}} \\ 0 & \text{otherwise} \end{cases}$$

Show that $\sqrt{n}(Y_n - \mu) \xrightarrow{d} Y^{\mu}$. Find the distribution of Y^{μ}.

3. Let X, X_n be a sequence of random variables on $(\Omega, \mathcal{B}, P)$. Show that $X_n \xrightarrow{d} X$ if and only if $E(\phi(X_n)) \to E(\phi(X))$ for all bounded continuous functions ϕ.

4. Two types of coin are produced at a factory: a fair coin and a biased one that comes up heads 55% of the time. We have one of these coins but do not know whether it is a fair or biased coin. In order to ascertain which type of coin we have, we shall perform the following statistical test. We shall toss the coin 1000 times. If the coin comes up heads 525 or more times we shall conclude that it is a biased coin. Otherwise, we shall conclude that it is fair. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased?

5. Let $a_n = \sum_{k=0}^{n} \frac{n^k}{k!} e^{-n}$, $n \geq 1$. Using the Central Limit Theorem evaluate $\lim_{n\to\infty} a_n$.

6. We wish to find the probability that a certain thumb tack will fall on its flat head when tossed. Suppose we know that $0.6 < p < 0.9$. How many trials are needed in order that we can be 95% sure that the observed frequency differs from p by less than $p/10$?