Problems to be turned in : **Due: Tuesday, April 1, 2010**

1. **(Slutsky's theorem)** Let $\{X_n, X, Y_n : n \in \mathbb{N}\}$ be random variables on a probability space $(\Omega, \mathcal{B}, P)$. Let $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$ where $c \in \mathbb{R}$ (i.e. $Y_n \xrightarrow{p} Y$ with $Y = c$ *a.e.*). Then
   (a) $X_n Y_n \xrightarrow{d} cX$
   (b) $\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{c}$ if $c \neq 0$

2. Let $X_n \xrightarrow{d} X$ then show that $X_n^2 \xrightarrow{d} X^2$.

3. Let $Y \stackrel{d}{=} N(0,1)$. Let $X_n = (-1)^n Y$. Discuss convergence a.e, in probability, and in distribution of X_n.

4. Let $Y_1, Y_2, \cdots, Y_n$ be independent random variables, each uniformly distributed over the interval $(0, \theta)$. Show that $\max\{Y_1, \cdots, Y_n\}$ converges in probability toward θ as $n \to \infty$.

5. Let $X_n \xrightarrow{d} X$ and let F denote the distribution function of X. Let a be continuity point of F. Show that $P(X_n = a) \to 0$.

6. Let $\{X_n : n \geq 1\}$ be a sequence of random variables that is monotonically increasing, i.e. $X_{n+1}(\omega) \leq X_n(\omega)$ for all $\omega \in \Omega$, $n \in \mathbb{N}$. If $X_n \xrightarrow{p} X$ then show that $X_n \xrightarrow{a.e.} X$.

7. Let Y_n be a sequence of independent and identically distributed (henceforth abbreviated to **i.i.d.**) random variables and let $X_n = \frac{Y_n}{n}$. Show that X_n converges in probability. Decide whether X_n converges a.e. or not.(Hint: use Borel-Cantelli lemma.)

8. Let X_n be a sequence of independent random variables on $(\Omega, \mathcal{B}, P)$, such that $X_n \stackrel{d}{=}$ Exponential (a_n) with $a_n = \ln(n+1)$. Show that the sequence converges to zero in probability. Does the sequence converge to zero almost everywhere ? (Hint: use Borel-Cantelli lemma.)

9. Let $\mathcal{F} = \{F : \mathbb{R} \to [0,1] : F \text{ is a distribution function.}\}$ Define the function $d : \mathcal{F} \times \mathcal{F} \to [0, \infty)$ by
$$d(F, G) = \inf\{\epsilon > 0 : G(x - \epsilon) - \epsilon \leq F(x) \leq G(x + \epsilon) + \epsilon\}.$$
Show that $(\mathcal{F}, d)$ is a metric space. Further show that a sequence of random variables $\{X_n\}$ converges in distribution to X if and only if $\rho(F_{X_n}, F_X) \to 0$.

10. Let $\mathcal{X}$ be the set of all random variables on the probability space $(\Omega, \mathcal{B}, P)$. Define a function $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by
$$\rho(X, Y) = E(\min(|X - Y|, 1),$$
for any $X, Y \in \mathcal{X}$. Show that $(\mathcal{X}, \rho)$ is a metric space. Further show that a sequence of random variables $\{X_n\}$ converges in probability to X if and only if $\rho(X_n, X) \to 0$.