Due: Tuesday, March 23rd, 2010

1. Let μ_1 and μ_2 be two probability measures on $(\Omega, \mathcal{B})$ and
$$\mathcal{C} = \{f : \mathbb{R} \to \mathbb{R} : f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-a}{\sigma})^2}, a, \sigma \in \mathbb{R}\}.$$
Suppose
$$\int f d\mu_1 = \int f d\mu_2, \ \forall f \in \mathcal{C}, \quad (1)$$
then $\mu_1 = \mu_2$. Is this fact true if μ_i are σ-finite measures ?

2. Verify the formulae for the characteristic functions given below:-

Distribution	**Characteristic Function** $\phi(t), t \in \mathbb{R}$
Bernoulli (p)	$1 - p + pe^{it}$
Binomial (n, p)	$(1 - p + pe^{it})^n$
Uniform $(\{1, 2, \ldots, n\})$	$\frac{e^{it}(1-e^{it})}{n(1-e^{int})}$
Poisson (λ)	$e^{\lambda}(e^{iu-1})$
Uniform (a, b)	$\frac{e^{ibt}-e^{iat}}{i(b-a)t}$
Normal (m, σ^2)	$e^{-imt-t^2\frac{\sigma^2}{2}}$

3. Let X be a real valued random variable on $(\Omega, \mathcal{F}, P)$ with characteristic function ϕ. Show that
   (a) ϕ is a bounded continuous function with $\phi(0) = 1$
   (b) If $E(|X|^m) < \infty$ for some positive integer m, then show that ϕ is m-times differentiable.

4. Let $n \in \mathbb{N}$ and X be an $\mathbb{R}^n$ valued random variable on $(\Omega, \mathcal{F}, P)$. Define its characteristic function to be
$$\phi_X(a) = E(e^{i<X,a>}),$$
where $a \in \mathbb{R}^n$ and $< X, a > (\omega) = \sum_{j=1}^n X_j(\omega)a_j$.
   (a) Generalise Theorem on uniqueness to such random vectors.
   (b) For any vector $\alpha \in \mathbb{R}^n$ and matrix $B \in M_{n\times n}(\mathbb{R})$ show that $\phi_{\alpha+BX}(a) = e^{i<\alpha,a>}\phi_X(B^T a)$, where B^T is the transpose of the matrix B. (Vectors are thought of as column vectors.)

(c) Suppose $X = (X_1, X_2, \ldots, X_n)$ where each $\{X_i : 1 \le i \le n\}$ is a real valued random variable on $(\Omega, \mathcal{B}, P)$. Then show that $\{X_i : 1 \le i \le n\}$ are independent if and only if

$$\phi_X(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} \phi_{X_i}(a_i),$$

where $a_i \in \mathbb{R}$, for $1 \le i \le n$.

5. Suppose $(\Omega, \mathcal{B}, \mu)$ is a σ -finite measure space and $f : \Omega \to \mathbb{R}$ is a non-negative $((\mathcal{B}, \mathcal{B}_{\mathbb{R}})-)$ measurable function. Define $\mathcal{G} = \{(w,t) \in \Omega \times \mathbb{R} : 0 \le t \le f(w)\}$. Show that $\mathcal{G} \in \mathcal{B} \otimes \mathcal{B}_{\mathbb{R}}$, and that $(\mu \times m)(\mathcal{G}) = \int f d\mu$, where m denotes Lebesgue measure on $\mathbb{R}$.

6. Let $\alpha > 0$, t, $a < b$ be real numbers and f be integrable on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, dx)$. Show that

$$\int_{[a,b]} f(\alpha x + t) dx = \int_{[\frac{a-t}{\alpha}, \frac{b-t}{\alpha}]} f(x) \alpha dx$$

7. Let $\Omega_1 = \Omega_2 = \mathbb{N}$, $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{P}(\mathbb{N})$. Define the measures μ_i on $(\Omega_i, \mathcal{B}_i)$ by $\mu_i(\{k\}) = 2^{-k}$. Define $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ by

$$f(m,n) = \begin{cases} -n2^{2n} & \text{if } m = n \\ n2^{2n} & \text{if } m = n-1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that f is $\mathcal{B}_1 \otimes \mathcal{B}_2$ measurable.
(b) Observe that

$$\int_{\Omega_2} \int_{\Omega_1} f(m,n) d\mu_1(m) d\mu_2(n) \neq \int_{\Omega_1} \int_{\Omega_2} f(m,n) d\mu_2(n) d\mu_1(m),$$

thereby emphasising the importance of integrability in the hypotheses of Fubini's theorem.

8. Let $I = [0,1]$. Let $I_1 = I_{11} = (\frac{1}{3}, \frac{2}{3})$ be the open middle third interval of I. Next, let $I_{21} = (\frac{1}{9}, \frac{2}{9})$ and $I_{22} = (\frac{7}{9}, \frac{8}{9})$ be the two open middle third intervals of $I - I_1$. Let $I_2 = I_{21} \cup I_{22}$. For $j \ge 3$ and $k = 1, 2, 3 \ldots, 2^{j-1}$, let I_{jk} be the open middle third intervals of $I - \cup_{k=1}^{j-1} I_k$ and let $I_j = \cup_{k=1}^{2^{j-1}} I_{jk}$. Finally, let $C = I - \cup_{j=1}^{\infty} I_j$. C is called the **Cantor set**.

(a) Show that C is compact and uncountable.

(b) Show that $\lambda(C) = 0$, where λ is lebesgue measure on $[0,1]$.

(c) Show that if $x \in C$ then $x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}$ where $a_j = 0$ or $a_j = 2$ for all j.

(d) Define a function $f : C \to [0,1]$ as: $f(x) = \sum_{j=1}^{\infty} \frac{b_j}{2^j}$, where $x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}$ and $b_j = \frac{a_j}{2}$.

9. Le C be as in previous problem.

(a) Show that f maps C onto $[0,1]$.

(b) If $x, y \in C$, $x < y$, and x, y are not the end points of one of the intervals removed from $[0,1]$ to obtain C, then $f(x) < f(y)$.

(c) If $x, y \in C$, $x < y$, and x, y are end points of one of the intervals removed from $[0,1]$ to obtain C, then show that $f(x) = f(y) = \frac{p}{2^k}$ for some $p, k \in \mathbb{N}$ and p not divisible by 3. (Hint: If x is an end point of one of the intervals removed to obtain C, then $x = \frac{p}{3^k}$ for some $p, k \in \mathbb{N}$ and p not divisible by 3. Use (1) and 2(a) to obtain the result.)

(d) Extend f to a map from $[0,1]$ onto itself by defining its value on each interval missing from C to be its value at the end points. Show that f is continuous but not absolutely continuous (Hint: $f' = 0$ *a.e.*).

10. Find the characteristic function of the Gamma distribution with parameters (n, λ).