Due: April 22nd, 2009
Problems to be turned in : 2,5

1. Let X_n be a Markov chain on state space $S = \{1, 2, 3, 4, 5, 6, 7\}$ with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

   Determine the closed communicating classes and their periodicity.

2. **(Random walk on a Circle)** Let S be $\{0, 1, 2, \ldots, L\}$. Consider a particle, starting at a uniformly chosen point in S. At each time step, the particle goes to the right with probability p and left with probability $1 - p$ with $0 < p < 1$. If it tries go left from 0 it jumps up to L and similarly if it tries to go right from L it jumps to 0.

   (a) Write down the transition probability matrix, when $L = 6$.

   (b) For general L, show that the chain is irreducible.

3. Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$. and $m \geq 1$, show that

$$P_i(N(j) \geq m) = P_i(T_j < \infty)(P_j(T_j < \infty))^{m-1}.$$

4. Let X_n be an irreducible Markov chain on a state space S with transition matrix $\mathbf{P}$. Then show that

$$\begin{aligned} P_i(T_i < \infty) = 1 &\iff i \in S \text{ is recurrent} \\ &\iff P_j(T_i < \infty) = 1 \ \forall j \in S. \end{aligned}$$

5. **(Random walk on $\mathbb{Z}$)** Let X_n be a Markov chain on $S = \mathbb{Z}$ with tranistion matrix $\mathbf{P}$ given by

$$p_{ij} = \begin{cases} p & \text{if } j = i + 1 \\ 1 - p & \text{if } j = i - 1 \\ 0 & \text{otherwise,} \end{cases}$$

   where $0 < p < 1$.

   (a) Show that the chain is irreducible and every state has period 2.

   (b) Show that for every $i \in \mathbb{Z}$ $n \in \mathbb{N}$, $p_{ii}^{2n} = \binom{2n}{n} p^n (1 - p)^n$.

   (c) Decide whether the chain is recurrent or transient.

   (d) What can you say if $p \in \{0, 1\}$?

6. **(Simple Symmetric Random walk on $\mathbb{Z}^d$)** Let $S = \mathbb{Z}^d$, and call i a neigbhour of j in $\mathbb{Z}^d$, denoted by $i \sim j$, if the Euclidean distance from i to j is one. Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ given by:

$$p_{ij} = \begin{cases} \frac{1}{2d} & \text{if } i \sim j \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Show that the chain is irreducible and determine its periodicity.

   (b) Decide whether the chain is recurrent or transient for the cases $d = 2, 3$.

   (c) What can you say for $d \geq 4$?

7. Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$.

   (a) For $i, j \in S$ show that

$$p_{ij}^n = \sum_{k=1}^{n} f_{ij}^k p_{jj}^{n-k}, \tag{0.1}$$

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   where $f_{ij}^n = P_j(T_i = n)$ for $i, j \in S$.

   (b) If j is transient then show that $\sum_{n=1}^{\infty} p_{ij}^n < \infty$ for all $i \in S$.

   (c) Suppose S is a finite set. Show that there is at least one recurrent state. (Hint: Assume that every state is transient, use the previous part to arrive at a contradiction)