Due: April 15th, 2009

Problems to be turned in : 3, 6

1. Suppose $\mathcal{A}$ is an algebra of subsets of a set Ω . Suppose $\mu : \mathcal{A} \to [0,1]$ is a finitely additive set function. The following conditions on μ are equivalent :

   (a) μ is countably additive on $\mathcal{A}$;

   (b) μ is 'continuous from above at the empty set' : i.e., if $A_n \in \mathcal{A}$, $A_1 \supseteq A_2 \supseteq \cdots$ and $\cap_{n=1}^{\infty} A_n = \phi$, then $\lim_{n\to\infty} \mu(A_n) = 0$.

2. Let $\Omega = \{0,1\}^{\mathbb{N}}$. Let $\pi_n : \Omega \to \Omega_n$ denote the projection onto the first n coordinates, where $\Omega_n = \{(w_1, \ldots, w_n) : w_i = 0 \text{ or } 1\}$. Let $\mathcal{B}_n = \{\pi_n^{-1}(E) : E \subseteq \Omega_n\}$ of all such n-dimensional cylinders is a finite σ-algebra of subsets of Ω.

   (a) Show that $\mathcal{A} = \cup_{n=1}^{\infty} \mathcal{B}_n$ is an algebra of subsets of Ω.

   (b) Let $\mathcal{B} = \sigma(\mathcal{A})$. Let μ be a probability measure defined on B. Define $\mu_n = \mu|\mathcal{B}_n$. Show that μ_n is a probability measure defined on $\mathcal{B}_n$, and the $\mu_n' s$ satisfy the consistency condition : $\mu_n = \mu_{n+1}|\mathcal{B}_n$.

   (c) Conversely, if μ_n is a probability measure defined on $\mathcal{B}_n$, and if $\mu_{n+1}|\mathcal{B}_n = \mu_n$, then we may define the set function μ on $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ by $\mu|\mathcal{B}_n = \mu_n$. Verify that μ is finitely additive on $\mathcal{A}$, deduce that μ is a well-defined countably additive set function, and conclude that μ extends uniquely to a probability measure defined on $\sigma(\mathcal{A}) = \mathcal{B}$.

   (d) Discuss the significance of the resulting measure μ in the special case when $\mu_n : \mathcal{B}_n \to [0,1]$ by $\mu_n\left(\pi_n^{-1}(E)\right) = |E|.2^{-n}$, where $|E|$ denotes the cardinality of the subset E of Ω_n.

   (e) In what sense can the function $T : \Omega \to [0,1]$ defined by $T(w) = \sum_{n=1}^{\infty} w_n 2^{-n}$, be regarded as an an 'isomorphism between the measure spaces' $(\Omega, \mathcal{B}, \mu_{Bernoulli})$ and $([0,1], \mathcal{B}_{[0,1]}, \mu_{Lebesgue})$?

3. Let $\Omega_0 = \{1, 2, \cdots, N\}, \mathcal{B}_0 = 2^{\Omega_0}$ and $\Omega = \Omega_0^{\mathbb{N}}$ (is the set of infinite sequences in Ω_0) and $\mathcal{B} = \otimes_{n=1}^{\infty} \mathcal{B}_0$.

   (a) Let P be a probability measure defined on $\mathcal{B}$. Define

   $$p_{i_1,\cdots,i_n} = P(\{w \in \Omega : X_1(w) = i_1, \cdots, X_n(w) = i_n\}) \tag{0.1}$$

   for arbitrary $n = 1, 2, \cdots$ and $i_1, \cdots, i_n \in \{1, 2, \cdots, N\}$. Then, for all $i_1, \cdots, i_n \in \{1, 2, \cdots, N\}$ and for all $n = 1, 2, \cdots$,

   $$\begin{aligned} p_{i_1,\cdots,i_n} &\geq 0; \\ \sum_{i_1,\cdots,i_n=1}^{N} p_{i_1,\cdots,i_n} &= 1; \\ p_{i_1,\cdots,i_n} &= \sum_{j=1}^{N} p_{i_1,\cdots,i_n,j}. \end{aligned} \tag{0.2}$$

   (b) Conversely, if $\{p_{i_1,\cdots,i_n} : n = 1, 2, \cdots, i_1, \cdots, i_n \in \{1, \cdots, N\}\}$ are arbitrary numbers satisfying equations (0.2) , then there exists a unique probability measure P on $(\Omega , \mathcal{B})$ satisfying equations (0.1).

4. The examples given below can be modeled by a Markov chain. Determine the state space, initial distribution and the transition matrix for each.

   (a) Suppose N black balls and N white balls are placed in two urns so that each urn contains N balls. At each step one ball is selected at random from each urn and the two balls interchange places. The state of the system at time $n \in \mathbb{N}$ is the number of white balls in the first urn after the n-th interchange.

   (b) Suppose a gambler starts out with a certain initial capital of N rupees and makes a series of 1 rupee bets against the gambling house until her captial runs out. Assume that she has probability p of winning each bet. Let the state of the system at time $n \in \mathbb{N}$ denote her capital at the n-th bet.

5. Meteorologist Chakrapani could not predict rainy days very well in the wet city of Cherapunjee. (*Reality:* In Cherapunjee it is actually really easy to predict rain. It only rains two months in a year, that too on every day of those two months!)

   So he decided to use the following prediction model for rain. If it had rained yesterday and today, then it will rain tomorrow with probability 0.5. If it rained today but not yesterday, then it will rain tomorrow with probability 0.3. If it did not rain today but had rained yesterday, then it will rain tomorrow with probability 0.1. Finally if it did not rain today and had not rained yesterday, then it will rain tomorrow with probability 0.1. Let X_n denote R if it rained on day $n \in \mathbb{N}$ and D if it was a dry day (no rain). Assume that with probability 0.5 it rains on day 0.

   Show that $\{X_n : n \geq 0\}$ is not a Markov chain but $(Y_n = (X_n, X_{n-1}), n \geq 1)$ is a Markov chain. Write down the state space, initial distribution and the transition matrix for the chain Y_n.

6. Consider a Markov chain X_n on state space $\{A, B, C\}$ with initial distribution μ and transition matrix given by

$$\mathbf{P} = \begin{pmatrix} .2 & .4 & .4 \\ .4 & .4 & .2 \\ .4 & .6 & 0 \end{pmatrix}.$$

   (a) what is the probability of going from state A to state B in one step ?

   (b) what is the probability of going from state B to state C in exactly two steps ?

   (c) what is the probability of going from state C to state A in exactly two steps ?

   (d) what is the probability of going from state C to state A in exactly three steps ?

   (e) Calculate the second, third and fourth power of this matrix. Do you have a guess for $\mathbf{P}^n$ for large n (You will find that $\lim_{n\to\infty} p_{ij}^n = \pi(j)$ for $i, j \in S$ for some vector $(\pi(j))_{j\in S}$ which is called the stationary distribution. This will be discussed later in this chapter.)