Due: March 25th, 2009
Problems to be turned in: 1,5

1. **(Slutsky's theorem)** Let $\{X_n, X, Y_n : n \in \mathbb{N}\}$ be random variables on a probability space $(\Omega, \mathcal{B}, P)$. Let $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$ where $c \in \mathbb{R}$ (i.e. $Y_n \xrightarrow{p} Y$ with $Y = c$ *a.e.*). Then
   (a) $X_n Y_n \xrightarrow{d} cX$
   (b) $\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{c}$ if $c \neq 0$
2. Construct an example where $X_n \xrightarrow{d} X$ such that there exists $x \in \mathbb{R}$ where $F_n(x) \not\to F(x)$.
3. Let $\mathcal{F} = \{F : \mathbb{R} \to [0,1] : F \text{ is a distribution function.}\}$ Define the function $d : \mathcal{F} \times \mathcal{F} \to [0, \infty)$ by
$$d(F,G) = \inf\{\epsilon > 0 : G(x-\epsilon) - \epsilon \leq F(x) \leq G(x+\epsilon) + \epsilon\}.$$
   Show that $(\mathcal{F}, d)$ is a metric space. Further show that a sequence of random variables $\{X_n\}$ converges in distribution to X if and only if $d(F_{X_n}, F_X) \to 0$.
4. Let $\mathcal{X}$ be the set of all random variables on the probability space $(\Omega, \mathcal{B}, P)$. Define a function $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by
$$\rho(X,Y) = E(\min(|X-Y|, 1),$$
   for any $X, Y \in \mathcal{X}$. Show that $(\mathcal{X}, \rho)$ is a metric space. Further show that a sequence of random variables $\{X_n\}$ converges in probability to X if and only if $\rho(X_n, X) \to 0$.
5. Let μ_1 and μ_2 be two probability measures on $(\Omega, \mathcal{B})$ and
$$\mathcal{C} = \{f : \mathbb{R} \to \mathbb{R} : f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-a}{\sigma})^2}, a, \sigma \in \mathbb{R}, \sigma \neq 0\}.$$
   Suppose
$$\int f d\mu_1 = \int f d\mu_2, \forall f \in \mathcal{C}, \tag{0.1}$$
   then $\mu_1 = \mu_2$.
6. Let $t \in \mathbb{R}$, $i = \sqrt{-1}$, and $n \in \{0\} \cup \mathbb{N}$. Then show that
$$|e^{it} - \sum_{k=0}^{n} \frac{it^k}{k!}| \leq \min(\frac{|t|^{n+1}}{n+1!}, 2\frac{|t|^n}{n!})$$