Due: March 11th, 2009
Problems to be turned in: 1,4

1. Suppose (X, Y) is a point chosen uniformly on the triangle $\{(x, y) : x \geq 0, y \geq 0, x + y = 4\}$. Find the conditional probability $P(Y > 1 \mid X = x)$.
2. Let X, X_n, $n \in \mathbb{N}$, be integrable random variables on a probability space,$(\Omega, \mathcal{B}, P)$ and $\mathcal{C}$ be a sub-σ algebra of $\mathcal{B}$.
   (a) Suppose $X_n, X \geq 0$ and $X_n \uparrow X$ on Ω. Then show that
   $$E(X_n \mid \mathcal{C}) \uparrow E(X \mid \mathcal{C}).$$
   (b) Suppose $X_n \to X$ such that there exists a integrable Y on $\mathcal{C}$ such that $\mid E(X_n \mid \mathcal{C}) \mid \leq Y \ \forall n$; then
   $$E(X_n \mid \mathcal{C}) \to E(X \mid \mathcal{C}).$$
3. Let X_i be i.i.d. Bernoulli(p) random variables. Define $S_n = \sum_{i=1}^n X_i$. For $1 \leq m \leq n$, find the conditional distribution of S_m given $S_n = k$. Have you seen this distribution before ? Compute $E(S_m|S_n = k)$ and $\mathrm{Var}(S_m|S_n = k)$.
4. The number of eggs laid by a certain kind of insect follows a Poisson distribution. It is known that two such insects have laid their eggs in a particular area. If the total number of eggs in the area is 150, what is the chance that the first insect laid at least 90 eggs ?
5. Suppose X is distributed uniformly on $(-1, 1)$ and given $X = x$, Y is distributed uniformly on $(-\sqrt{1 - x^2}, \sqrt{1 - x^2})$. What is the joint distribution of (X, Y) ?
6. Let X be a real valued random variable on $(\Omega, \mathcal{F}, P)$ with characteristic function ϕ. Show that
   (a) ϕ is a bounded continuous function with $\phi(0) = 1$
   (b) If $E(|X|^m) < \infty$ for some positive integer m, then show that ϕ is m-times differentiable.
7. Let $n \in \mathbb{N}$ and X be an $\mathbb{R}^n$ valued random variable on $(\Omega, \mathcal{F}, P)$. Define its characteristic function to be
   $$\phi_X(a) = E(e^{i<X,a>}),$$
   where $a \in \mathbb{R}^n$ and $< X, a > (\omega) = \sum_{j=1}^n X_j(\omega) a_j$.
   (a) Generalise Theorem (on Uniqueness) to such random vectors.
   (b) For any vector $\alpha \in \mathbb{R}^n$ and matrix $B \in M_{n\times n}(\mathbb{R})$ show that $\phi_{\alpha+BX}(a) = e^{i<\alpha,a>}\phi_X(B^T a)$, where B^T is the transpose of the matrix B. (Vectors are thought of as column vectors.)
   (c) Suppose $X = (X_1, X_2, \ldots, X_n)$ where each $\{X_i : 1 \leq i \leq n\}$ is a real valued random variable on $(\Omega, \mathcal{B}, P)$. Then show that $\{X_i : 1 \leq i \leq n\}$ are independent if and only if
   $$\phi_X(a_1, a_2, \ldots, a_n) = \prod_{i=1}^n \phi_{X_i}(a_i),$$
   where $a_i \in \mathbb{R}$, for $1 \leq i \leq n$.
8. Find the characteristic function of the Gamma distribution with parameters (n, λ).
9. Verify the formulae for the characteristic functions given in the Table below:-

Distribution	**Characteristic Function** $\phi(t), t \in \mathbb{R}$
Bernoulli (p)	$1 - p + pe^{it}$
Binomial (n, p)	$(1 - p + pe^{it})^n$
Uniform $(\{1, 2, \ldots, n\})$	$\frac{e^{it}(1-e^{it})}{n(1-e^{int})}$
Poisson (λ)	$e^{\lambda}(e^{iu-1})$
Uniform (a, b)	$\frac{e^{ibt}-e^{iat}}{i(b-a)t}$
Normal (m, σ^2)	$e^{-imt-t^2\frac{\sigma^2}{2}}$

Table 1: Characteristic Functions of Standard Distributions