Due Date: February 18th, 2009
Problems to be turned in: 1

1. A person consumes three commodities. Suppose the utility function was given by:

$$u(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = x_1^{\frac{1}{3}} + \min(x_2, x_3).$$

Assume that the person has a postive income I, and (positive) prices of the commodities are p_1, p_2, p_3. Is there a solution to the utilitiy maximisation problem ? If yes then can you use Kuhn-Tucker to characterise the solutions ?

2. A firm produces a single output y using three inputs x_1, x_2, x_3 in non-negative quantities through the relationship

$$y = g(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = x_1(x_2 + x_3).$$

The unit price of y is $p_y > 0$, while that of the input x_i is $w_i > 0$, $i = 1, 2, 3$.

(a) Describe the firm's profit maximisation problem, and derive the equations that define the critical points of the Lagrangean L in this problem.

(b) Show that the Lagrangean L has multiple critical points for any choice of (p_y, w_1, w_2, w_3).

(c) Show that none of these critical points identifies a solution of the profit-maximisation problem. Explain why.

3. Consider the following problem:

$$\begin{array}{ll} \text{minimise} & \sum_{i=1}^{n} a_i e^{-x_i} \\ \text{subject to} & \sum_{i=1}^{n} x_i = b \\ & x_i \geq 0,\ i = 1, 2, \ldots, n, \end{array}$$

where $a_1 \geq a_2 \geq \ldots \geq a_n > 0$ and $b > 0$. Show that in the optimal solution $x_1 \geq x_2 \geq \ldots \geq x_n > 0$ and that $x_j > 0$ if and only if $a_1 a_2 \ldots a_j < e^b (a_j)^j$

4. Suppose a_1, a_2, a_3, b are given positive numbers. Define $\phi(b)$ as the solution to the following problem:

$$\begin{array}{ll} \text{minimise} & \sum_{i=1}^{3} x_i \\ \text{subject to} & \sum_{i=1}^{3} \frac{x_i^2}{a_i} = b \\ & x_i \geq 0,\ i = 1, 2, 3. \end{array}$$

Derive an expression for $\phi(b)$ with appropriate justification. If b is changed to $b + \delta b$, compute the change in ϕ upto first order in δb.

5. Define $\phi(b)$ as the solution to the following problem:

$$\begin{array}{ll} \text{minimise} & \sum_{i=1}^{n} \frac{v_i}{x_i} \\ \text{subject to} & \sum_{i=1}^{n} a_i x_i \leq b \\ & x_i > 0,\ i = 1, 2, \ldots, n, \end{array}$$

where $a_i > 0, v_i > 0$ for $i = 1, 2, \ldots, n$ and $b > 0$.

Derive an expression for $\phi(b)$ with appropriate justification. If b is changed to $b + \delta b$, compute the change in ϕ upto first order in δb.

6. Find a point in $\mathbb{R}^3$ at which a plane parallel to $x + 2y + z = 0$ is tangent to the unit sphere.