Some key Elements of Game Theory

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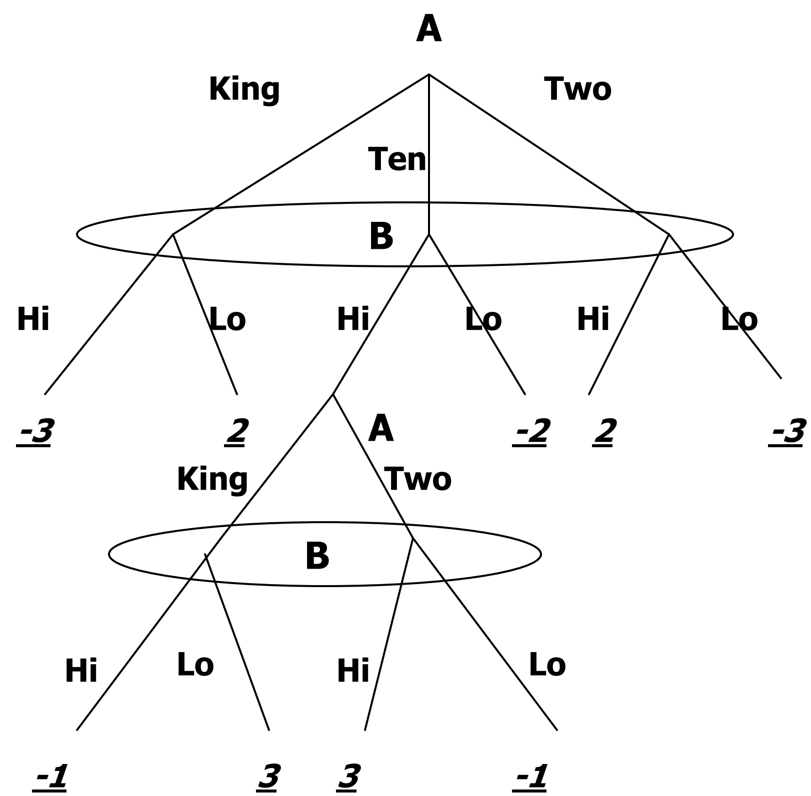
Agenda

- Homework - solution
- Saddle point
- Mixed strategies
- Minimax theorem
- Dominance of strategies
- Nash equilibrium and dominance
- Homework

Homework - Constructing an extensive form

- The game has 3 cards – King, Ten, and Two
- Player A chooses one of the three cards and puts it face down
- Player B guesses and calls either "High" or "Low"
- If she (B) is right (i.e. the card is "High" – king or if the card is "Low" – Two) then she wins Rs 3 from A but if she is wrong then she loses Rs 2 to A.
- If she calls "Low" and the card is Ten then she wins Rs 2
- If she calls "High" and the card is Ten then A has to choose between the king and Two and then put his choice down
- Now B has to again call either "High" or "Low". If she is right then she wins Rs 1 but if she is wrong then she loses Rs 3



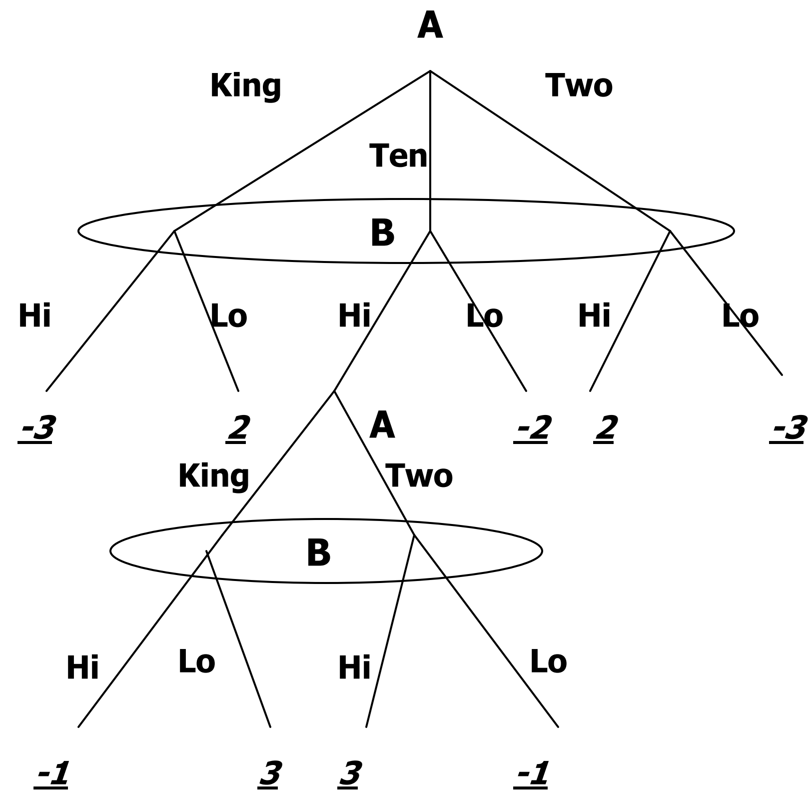
Homework - Identifying pure strategies

Pure strategies for A

Strategy	Description
A1	Choose king
A2	Choose ten and then if needed choose king
A3	Choose ten and then if needed choose two
A4	Choose two

Pure strategies for B

Strategy	Description
B1	Choose Hi and then if needed choose Hi
B2	Choose Hi and then if needed choose Lo
B3	Choose Lo



Homework - Creating a normal form

- With a complete list of pure strategies available, any particular complete play of the game can be represented as a combination of one strategy from each player's list
- Since a particular pure strategy for A and one for B uniquely determine a play and hence its outcome we can construct a matrix representing the outcomes

Pure strategies for A

Strategy	Description
A1	Choose king
A2	Choose ten and then if needed choose king
A3	Choose ten and then if needed choose two
A4	Choose two

Pure strategies for B

Strategy	Description
B1	Choose Hi and then if needed choose Hi
B2	Choose Hi and then if needed choose Lo
B3	Choose Lo

A/B	B1	B2	B3
A1	-3	-3	2
A2	-1	3	-2
A3	3	-1	-2
A4	2	2	-3

Saddle points

- Consider this simple game - Player A chooses a number (a) from the set {-1,0,1} and then player B (not knowing A's choice) also chooses a number (b) from the same set. After the choices are revealed, player B pays player A according to the formula a(b-a) + b(a+b) (such a function of payoff is called the payoff kernel)
- We can easily construct the payoff matrix

A/B	1 (b = -1)	2 (b = 0)	3 (b = 1)
1 (a = -1)	2	-1	-2
2 (a = 0)	1	0	1
3 (a = 1)	-2	-1	2

Saddle points

A/B	1 (b = -1)	2 (b = 0)	3 (b = 1)
1 (a = -1)	2	-1	-2
2 (a = 0)	1	0	1
3 (a = 1)	-2	-1	2

- A wishes to get the highest payoff. If he chooses 1 then B may choose -1 and then A will lose 2. Similarly if he choose -1 then B may choose 1. But if A chooses 0 then in the worst case scenario he gains or loses nothing. Similarly if B chooses 0 then her worst payoff is 0. For any other choice she may lose 2.
- If both players play this way, they will not regret their choice once the opponent's choice is known. Because they reason that given the opponent's choice they would have done worse by choosing differently.

Saddle points

- Such a case arises because in the payoff matrix there is an entry which is smallest in its row AND largest in its column. So for a general matrix game if g_{ij} is the entry in the i^{th} row and j^{th} column then we define
  - Pure maximin = $\max_{1 <= i <= m} (\min_{1<=j<= n} (g_{ij}))$
  - Pure minimax = $\min_{1 <= j <= m} (\max_{1<=i<= n} (g_{ij}))$
  - So for this example Pure maximin = Pure minimax

A/B	1 (b = -1)	2 (b = 0)	3 (b = 1)	Row min
1 (a = -1)	2	-1	-2	-2
2 (a = 0)	1	0	1	0
3 (a = 1)	-2	-1	2	-2
Column max	2	0	2	

- Such an entry for a particular i and j is called a *saddle point* and the entry itself is called the *value* of the game and the pair of pure strategies leading to it are called *optimal pure strategies*

Mixed strategies

- Not all matrix games have saddle points. For example the homework problem has a maximin = -2 and minimax =2 and hence no saddle point. In terms of pure strategies this game has no solution and no value. The most we can say (we may see the proof in later classes) Pure maximin <= Pure minimax
- So the question is how do we get a solution for such games. The approach we take is somewhat similar to the one used in theory of equations. $X^2 + 1 = 0$ has no solution in real numbers. This led to the invention of complex numbers and then the real numbers were regarded as a particular subset of complex numbers. Similarly we will generalize the concept of pure strategy.

Mixed strategies

- Consider this simple game. There is no saddle point and hence no solution in pure strategies. So ho does the player choose? In fact premature disclosure of a pure strategy will be a disadvantage for either player. A way out is to choose between pure strategies regulated by chance. Such a probability combination of the original pure strategies is called *mixed strategy*.

A/B	B1	B2
A1	1	-1
A2	-1	1

Mixed strategies

- In general if player A has m pure strategies then a mixed strategy for A consists of an ordered m-tuple such that

$$(x_1, \ldots, x_m), 0 \leq x_i \leq 1, \sum_{i=1}^{m} x_i = 1$$

  Where x_i denotes the probability that A will select the i^{th} pure strategy. Similar generalization can be made for player B

- If A uses mixed strategy x = $(x_1, \ldots, x_m)$ and B uses y = $(y_1, \ldots, y_n)$ in a game with payoff matrix g_{ij} then the expected payoff to A is given by (Homework – show this result)

$$P(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i g_{ij} y_j$$

- Having established the above, we can now look at pure strategy as a special kind of a mixed strategy. So an i^{th} pure strategy of A is basically a mixed strategy of (0,0,…1,0,..0) where 1 is in the i^{th} position.

Mixed strategies

- If B is made to announce in advance a mixed strategy of y_0 then A will choose x_0 such that

$$P(x_0, y_0) = \max_{x \in X} P(x, y_0)$$

- Under these circumstances the best that B can do is to announce y_0 so that

$$\max_{x \in X} P(x, y_0) = \min_{y \in Y} \max_{x \in X} P(x, y) = \bar{v} = \min i \max$$

- Similarly if we make A announce in advance then we get an expression of what is the least that A can expect to win (maximin) (Homework)

Minimax theorem and its use

- Von Neumann (in 1928) showed that (See the book by Tijs) for any matrix game maximin = minimax (say v) and any such game will have a solution consisting of
  - An optimal mixed strategy which ensures A an expected gain of at least V
  - An optimal mixed strategy which ensures B an expected loss of at most V
  - The value of V itself

Minimax theorem and its use

- If we think we have a mixed strategy solution to a matrix game we can easily verify it. Now we return to the homework problem. Let's see if A's mixed strategy of (1/2,0,0,1/2) and B's mixed strategy of (1/4,1/4,1/2) for B gives a solution of -1/2.
  - To verify this we first compute A's expectations (using his mixed strategies) against each of B's pure strategies.
  - Since the assumes solution is -1/2 in each of the above cases the payoff for A should be at least -1/2
  - Now we repeat the process to compute B's expectations (using her mixed strategies) against each of A's pure strategies and compare the payoff

A/B	B1 (1/4)	B2 (1/4)	B3 (1/2)	B's losses
A1 (1/2)	-3	-3	2	- 1/2
A2 (0)	-1	3	-2	- 1/2
A3 (0)	3	-1	-2	- 1/2
A4 (1/2)	2	2	-3	- 1/2
A's gains	- 1/2	- 1/2	- 1/2	

Dominance of strategies

- Consider the following game

A/B	B1	B2	B3
A1	1	-1	2
A2	-1	1	3
A3	-3	-2	4

- Player B who wishes to minimize the payoff for A realizes that B3 is an undesirable strategy compared with B1 since every corresponding payoff for B1 is better with B1. Thus whatever A does B1 will always give better results that B3 to B. We say that B1 dominates B3

Dominance of strategies

- In general for a 2-person zero sum game in normal form let X and Y denote the set of strategies (pure or mixed) for A and B respectively. Suppose that if A chooses x and y (belonging to X and Y) and the payoff to A is P(x, y) then
  - For player A $x_1 \in X$ dominates $x_2 \in X$ if $P(x_1, y) \geq P(x_2, y) \forall y \in Y$
  - For player B $y_1 \in Y$ dominates $y_2 \in Y$ if $P(x, y_1) \leq P(x, y_2) \forall x \in X$
  - The dominance is said to be strict if the corresponding inequality is strict for all choices of the opponent's strategy

Back to prisoners' dilemma

- We return to the prisoners' dilemma example. This is a non-zero sum game but one can still apply the concept of dominance to it.
- To get clarity, for the time being let's split the original game into two equivalent games – one from A's point of view and one from B's point of view. Let's study the payoffs

	Confess	Refuse
Confess	(5, 5)	(0, 10)
Refuse	(10, 0)	(2, 2)

	Confess	Refuse
Confess	5	0
Refuse	10	2

	Confess	Refuse
Confess	5	10
Refuse	0	2

Back to prisoners' dilemma

- Now let's look at the original game.
  - Player 1, can see that the payoffs in each cell of the top row are higher than each corresponding cell of the bottom row. So it can never be rational for her to play bottom-row strategy regardless of what her opponent does. We can simply delete the bottom row from the matrix.
  - Now it is obvious that Player 2 will not refuse to confess, since his payoff from confessing in the two cells that remain is higher than his payoff from refusing. So, once again, we can delete the one-cell column on the right from the game.
  - We now have only one cell remaining, that corresponding to the outcome brought about by mutual confession.

	Confess	Refuse
Confess	(5, 5)	(0, 10)
Refuse	(10, 0)	(2, 2)

	Confess	Refuse
Confess	5	0
Refuse	10	2

	Confess	Refuse
Confess	5	10
Refuse	0	2

Nash Equilibrium and Dominance

- A set of strategies is a Nash Equilibrium (NE) when no player can improve his payoff, given the strategies of all other players in the game, by changing his strategy (unilaterally).
- Thus we can say that no strategy can be NE if it is strictly dominated (because if it is dominated that means the player can unilaterally change his strategy for a better payoff)
- In a zero-sum games if A is playing a strategy such that, given B's strategy, she can't do any better, and if B is also playing such a strategy, then, since any change of strategy by A would have to make B worse off and vice-versa, it follows that this game can have no solution compatible with the players' mutual rationality other than its unique NE.

Non-zero Sum Games & Nash Equilibrium

- What can we say about the NE of this non-zero sum game?

A/B	B1	B2
A1	(10, 10)	0
A2	0	(1, 1)

  - Note that no rows or columns are strictly dominated here. But if Player A is playing A1 then Player B can do no better than B1, and vice-versa; and similarly for the A2-B2 pair.
  - Thus there are 2 NE for this game: (A1, B1) and (A2, B2)
  - If NE is the only solution concept, then we shall be forced to say that either of these outcomes is equally persuasive as a solution. However surely rational players with perfect information would converge on A1-B1? (Note that this is not like the situation in the Prisoners' Dilemma, where the socially superior situation is unachievable because it is not a NE. In this case both players have every reason to try to converge on the NE in which they are better off.)
  - This illustrates the fact that NE is a relatively (logically) weak solution concept, often failing to predict intuitively sensible solutions.

Homework

- Develop the extensive form of the game of matching pennies. In the game each player chooses to show heads or tails.
- Player A wins a Rupee if both players make the same choice and B wins a Rupee if the choices are different.
- List pure strategies of both the players and then give the normal form.
- Now the players decide to modify the game. The basic rule remains the same (i.e. A wins for the same choices while B wins for the different ones) but now a player has to win 2 out of 3 such mini games before he wins a Rupee.
- Modify the extensive form, pure strategies and the normal form.

Homework

- Represent the following game in extensive form and then reduce it to matrix form.
- Player A has an Ace and a Queen. Player B has a King and a Joker. The rank precedence is Ace > King > Queen but Joker is peculiar as it is described further.
- Each player contributes a Rupee to the pot before the game starts. Each selects one of the cards and reveals them simultaneously.
- If B selects the king then the highest card owner wins the pot and the game ends. If B selects the joker and A Queen then they share the pot equally and the game ends. If B selects a joker and A the Ace then A may either resign (in which case B gets the pot) or may demand a replay. If there is a replay then each of them put another Rupee in the pot. Now if B selects the joker and A the Ace then B wins the pot.