Introduction to Game Theory

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Agenda

- Conflicts, conflicts, and conflicts
- Game theory - Modeling conflicts
- Assumptions in game theory
- Philosophical and historical perspective
- A classical game – Prisoners’ dilemma
- Mathematical representation of games
- Homework

Conflicts, conflicts, and conflicts

- When do conflicts arise?
  - Such a situation exists when two or more decision makers who have different objectives act on the same system or share the same resources.
- Examples
  - Two people playing chess and contemplating their next moves
  - Congress and BJP deciding on the election agenda
  - India and Pakistan discussing the border issues
  - Two armies deciding on a strategy to attack each other

Game theory – for modeling the conflicts

- Game theory is a branch of mathematical analysis developed to study decision making in conflict situations.
- It is an interdisciplinary approach – mathematics and economics.
- Game theory was founded by the great mathematician John von Neumann. He developed the field with the great mathematical economist, Oskar Morgenstern.

Assumptions in Game theory

- Each decision maker (called player) has available to him two or more well-specified choices or sequences of choices (called strategy).
- Every possible combination of strategies available to the players leads to a well-defined end-state (win, loss, or draw) that terminates the game.
- A specified payoff for each player is associated with each end-state

Assumptions (Continued)

- Each decision maker has perfect knowledge of the game and of her opposition; that is, she knows in full detail the rules of the game as well as the payoffs of all other players.
- All decision makers are rational; that is, each player, given two alternatives, will select the one that yields her the greater payoff.

Game theory – a philosophical motivation

- Plato, in *Republic*, presents the following situation.
  - A soldier waiting for the battle may think that if his side is likely to defend successfully, then his contribution will not be necessary. But if he stays then he risks being killed/wounded.
  - On the other hand, if the enemy is going to win, then his chances of death/injury are higher and quite clearly serve no purpose. So it would appear that he is better off running away regardless of who is going to win the battle.
  - Now if all of the soldiers reason this way (as they probably will and should) then this will certainly bring about the outcome in which the battle is lost.

Game theory – a historical motivation

- Long before game theory came into its formal existence, its essence had occurred to some military leaders and it influenced their strategies.
- The Spanish conqueror Cortez was landing in Mexico with a small force. His soldiers had a good reason to fear their capacity to defeat the far more numerous Aztecs. So they were thinking on the same lines as we saw earlier.

Game theory – a historical motivation

- He removed the risk of his troops running away by burning the ships on which they had landed! Now the Spanish soldiers had to stand and fight.
- The Aztecs had seen the ships burning so they then reasoned as follows: Any leader who willfully destroys his own ships must have good reasons for such extreme optimism. Hence it must not be wise to attack an opponent who is sure that he can't lose (for whatever reason it may be). The Aztecs therefore retreated so Cortez's victory was bloodless!

Prisoners’ dilemma

- The police have arrested two people whom they know have committed an armed robbery together. But they don’t have enough evidence to convict. They do, however, have enough evidence to send each prisoner away for two years for the theft of the “getaway” car.
- The inspector has the following offer to each prisoner:
  - If you confess to the robbery, implicating your partner, but she does not confess, then you'll go free and she'll get ten years.
  - If you both confess, you'll each get 5 years.
  - If neither confesses, then each gets two years for the car theft.

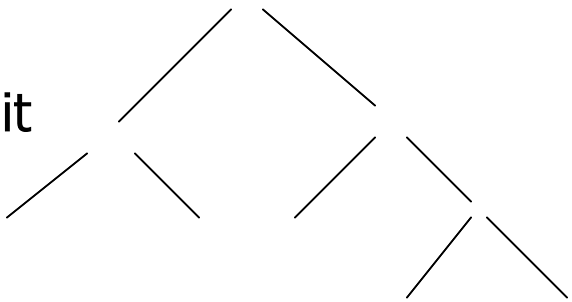
Let's play prisoners' dilemma!

		Player 2: Confess	Player 2: Refuse
Player 1	Confess	(5,5)	(0,10)
	Refuse	(10,0)	(2,2)

- Now you (player 1) play this with your neighbour (player 2) in two ways:
  - ➔ Each of you decide your strategy (either to confess or to refuse) simultaneously without talking to each other. Write your choice.
  - ➔ Now discuss and decide amongst yourselves what each of you should do and then write your choice on the sheet

Math. representation of games – Extensive form

- When a game is described fully (with its rules and pay-offs) we can abstract it to an "extensive" form. In this abstraction we are eliminating all the features which refer to the means of playing it (e.g. cards, dice, etc.)
- The conventional way of representing a game in a extensive form is to use "game trees". A game tree is said to represent a game in an extensive form *iff* this tree faithfully reproduces
  - each possible *state* together
  - with the possible *decisions* leading from it
  - and each possible outcome



Math. representation of games – Extensive form

- What can we say about *state* and how is it determined?
  - A state is determined not only by the current position but also by the unique history of play which led to the position
  - Thus in game theory, if the same position reached in a number of different ways, then they are regarded as different states

Math. representation of games – Normal form

- Games are sometimes represented in matrices. This is called the Normal form (also "Strategic form"). Matrices simply show the outcomes for every possible combination of strategies the players might use.
- The extensive-form contains information about sequences of play and levels of information about the game structure but the normal-forms do not.
- So a normal-form game could represent any one of several extensive-form games, hence in general a normal-form game is considered as a set of extensive-form games.

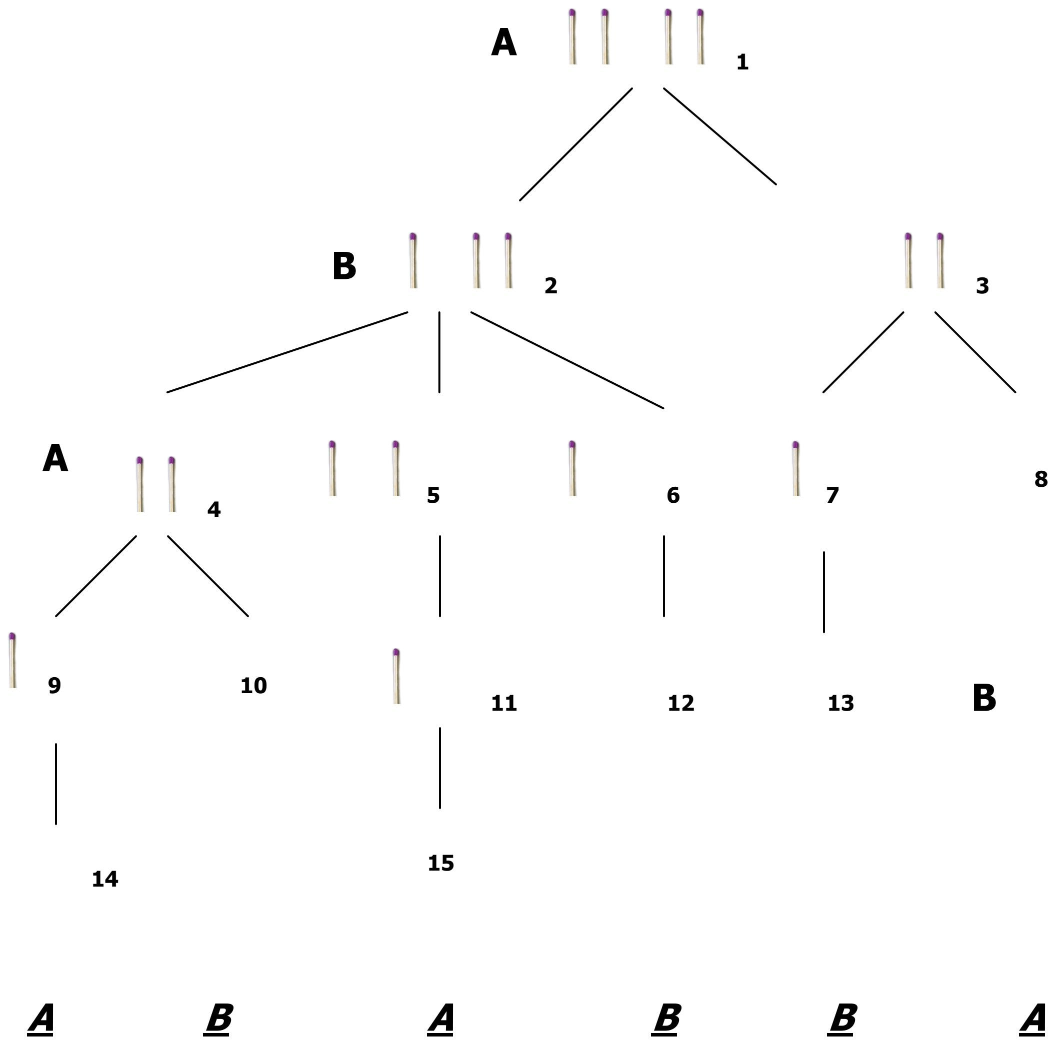
		Player 2	
		Confess	Refuse
Player 1	Confess	(5,5)	(0,10)
	Refuse	(10,0)	(2,2)

Which representation of games should be used?

- When do we prefer to study a game in its normal form?
  - When order of play is irrelevant to a game's outcome then normal form should be studied (since you are interested in the whole set)
  - Where order of play is relevant, then extensive form should be used (else your conclusions will be unreliable)

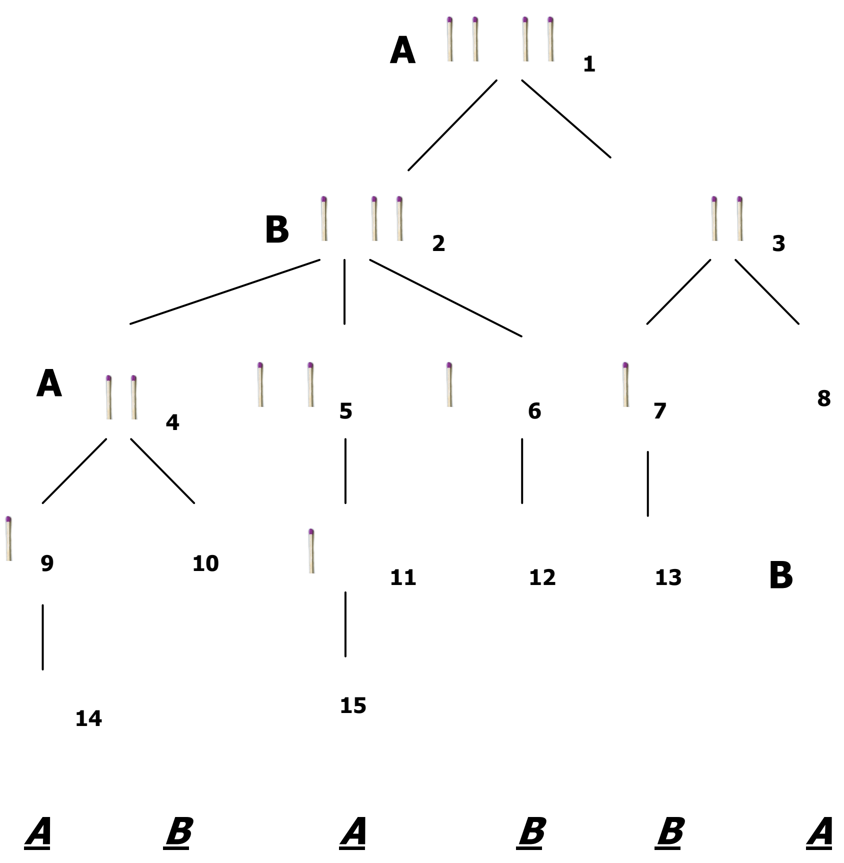
Constructing an extensive form – 2x2 Nim

- Four matches are set in 2 piles of 2 matches each
- Two players (A and B) take alternate turns
- At each turn the player selects a pile that has at least one match. From this pile she removes at least one. This continues until all the matches are gone.
- The player who removes the last match loses. The loser pays the winner Rs 1



Identifying pure strategies

- A pure strategy is a prior, comprehensive statement of choice to be made at each decision point which the player might possibly meet



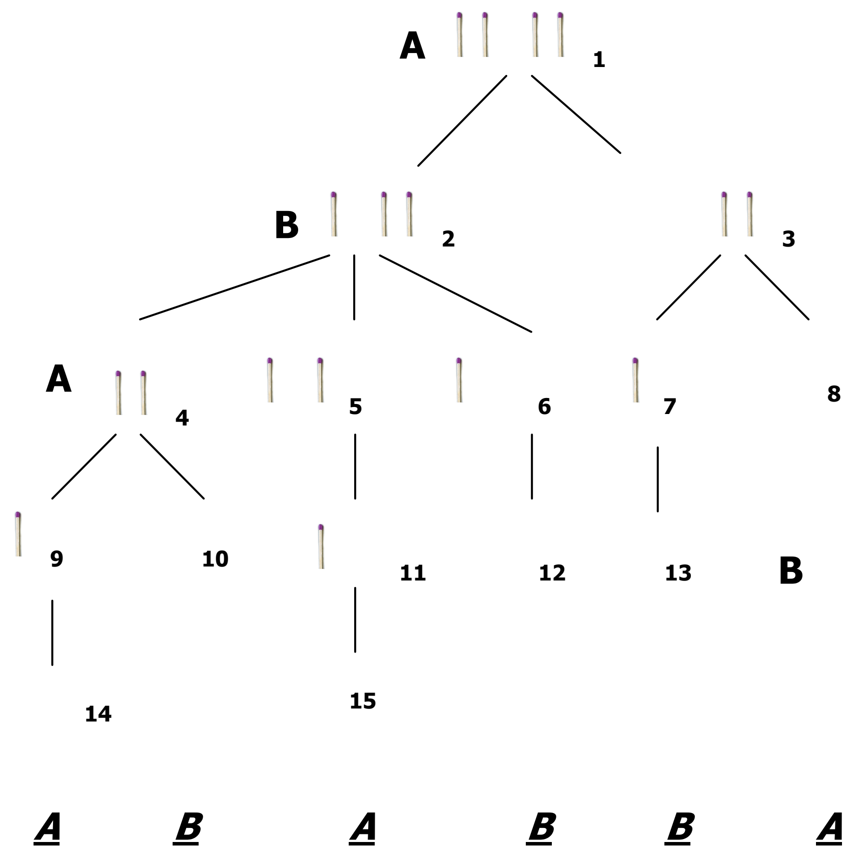
Pure strategies for A

Strategy	First turn	Second turn	
		if at	Then go to
A1	1 to 2	4	9
A2	1 to 2	4	10
A3	1 to 3	-	-

Identifying pure strategies

Pure strategies for B

Strategy	First turn	
	if at	Then go to
B1	2	4
	3	7
B2	2	5
	3	7
B3	2	6
	3	7
B4	2	4
	3	8
B5	2	5
	3	8
B6	2	6
	3	8



Creating a normal form

- With a comprehensive list of pure strategies available, any particular complete play of the game can be represented as a combination of one pure strategy from each player's list
- Thus from the adjoining tables we can formulate games such as
  - A1 and B1 yield "1 to 2 to 4 to 9 to 14"
  - A2 and B1 yield "1 to 2 to 4 to 10" and so on

Pure strategies for A

Strategy	First turn	Second turn	
		if at	Then go to
A1	1 to 2	4	9
A2	1 to 2	4	10
A3	1 to 3	-	-

Pure strategies for B

Strategy	First turn	
	if at	Then go to
B1	2 3	4 7
B2	2 3	5 7
B3	2 3	6 7
B4	2 3	4 8
B5	2 3	5 8
B6	2 3	6 8

Creating a normal form

- Since a particular pure strategy for A and one for B uniquely determine a play and hence its outcome we can construct a matrix representing the outcomes

A \ B	1	2	3	4	5	6
1	A (14)	A (15)	B (12)	A (14)	A (15)	B (12)
2	B (10)	A (15)	B (12)	B (10)	A (15)	B (12)
3	B (13)	B (13)	B (13)	A (8)	A (8)	A (8)

- Let’s now replace the outcomes with payoffs in the matrix. This is the matrix form or normal form or strategic form

A \ B	1	2	3	4	5	6
1	1	1	-1	1	1	-1
2	-1	1	-1	-1	1	-1
3	-1	-1	-1	1	1	1

Homework

- For the following 3-card game construct an extensive form using a game tree, identify pure strategies, and construct a normal form.
  - The game has 3 cards – King, Ten, and Two
  - Player A chooses one of the three cards and puts it face down
  - Player B guesses and calls either “High” or “Low”
  - If she (B) is right (i.e. the card is “High” – king or if the card is “Low” – Two) then she wins Rs 3 from A but if she is wrong then she loses Rs 2 to A.
  - If she calls “Low” and the card is Ten then she wins Rs 2
  - If she calls “High” and the card is Ten then A has to choose between the king and Two and then put his choice down
  - Now B has to again call either “High” or “Low”. If she is right then she wins Rs 1 but if she is wrong then she loses Rs 3