We will need the following two characterisations of continuity. The proofs of which are elementary and have been done in class.

Proposition 1 *Let $\Phi : \Theta \to P(S)$ be compact valued correspondence. Then, Φ is usc at $\theta \in \Theta$ if and only if for all sequences $\theta_k \to \theta \in \Theta$ and for all sequences $s_k \in \Phi(\theta_k)$, there is a subsequence $s_{k_l} \to s$ for some $s \in \Phi(\theta)$.*

Proposition 2 *Let $\Phi : \Theta \to P(S)$ be any correspondence. Suppose Φ is lower semi continuous at θ and $s \in \Phi(\theta)$. Then for all sequences $\theta_m \to \theta$, there is a sequence $s_m \in \Phi(\theta_m)$ and $s_m \to s$*

Theorem 1 *(**Maximum Theorem**) Let $S \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^l$. Let $f : S \times \Theta \to \mathbb{R}$ be a continuous function, and $\mathcal{D} : \Theta \to P(S)$ be a compact-valued, continuous correspondence. Let $f^* : \Theta \to \mathbb{R}$ and $\mathcal{D}^* : \Theta \to P(S)$ be defined by*

$$\begin{aligned} f^*(\theta) &= \max\{f(x,\theta) \mid x \in \mathcal{D}(\theta)\} \\ \mathcal{D}^*(\theta) &= \{x \in \mathcal{D}(\theta) \mid f(x,\theta) = f^*(\theta)\} \end{aligned}$$

Then f^* is a continuous function on Θ and $\mathcal{D}^*$ is a compact-valued, upper-semicontious correspondence on Θ.

Proof. Let $\theta_m, \theta \in \Theta$, so that $\theta_m \to \theta$. Let $x_m \in \mathcal{D}^*(\theta_m)$, which implies that $x_m \in \mathcal{D}(\theta_m)$. As $\mathcal{D}$ is continuous (in particular usc) and compact valued, Proposition 1 will imply that there exists a subsequence $x_{m_k} \to x$ for some $x \in \mathcal{D}(\theta)$.

Claim: $f(x,\theta) = f^*(\theta)$.

Proof of Claim: Suppose there is a $z \in \mathcal{D}(\theta)$ such that $f(z,\theta) > f(x,\theta)$.

$\mathcal{D}$ is lower semi-continuous at θ and $\theta_{m_k} \to \theta$, $z \in \mathcal{D}(\theta)$. So by Proposition 2 we have that $\exists$ $z_{m_k} \to z$. Therefore by continuity of f, we have that

$$\lim_{k\to\infty} f(z_{m_k}, \theta_{m_k}) = f(z,\theta) > f(x,\theta) = \lim_{k\to\infty} f(x_{m_k}, \theta_{m_k})$$

Consequently for large enough k, we have that $f(z_{m_k}, \theta_{m_k}) > f(x_{m_k}, \theta_{m_k})$. This is a contradiction to the fact that $x_{m_k} \in \mathcal{D}^*(\theta_{m_k})$. □

Once we have the claim, the proof follows easily:

(a) Then by continuity of f, $f^*(\theta_{m_k}) = f(x_{m_k}, \theta_{m_k}) \to f(x,\theta) = f^*(\theta)$. Since the above argument can be repeated for any subsequence of θ_m we have effectively shown that every subsequence of $f^*(\theta_m)$ has a further subseqence that converges to $f^*(\theta)$. This implies continuity of f^*.

(b) We have established that for every $\theta_m \to \theta$, and for any sequence $x_m \in \mathcal{D}^*(\theta_m)$, it has a subsequence that converges to x which is in $\mathcal{D}^*(\theta)$ by the claim. Hence $\mathcal{D}^*$ is upper semicontinous by the characterisation proven in class. $\square$