Theorem 1 **(Kuhn and Tucker)** *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $1 \le i \le l$, $h_i : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions. Suppose x^* is a local maximum of f on*

$$D = U \cup \{x \in \mathbb{R}^n : h_i(x) \ge 0, i = 1, \dots l\},$$

where U is an open set in $\mathbb{R}^n$. Let $E \subset \{1, \dots, l\}$ denote the set of effective constraints at x^* and let $h_E = (h_i)_{i \in E}$. Suppose rank($Dh_E(x^*)$) $=| E |$, then there exists a vector $\lambda^* = (\lambda_1^*, \dots, \lambda_l^*)$, $\lambda_i \in \mathbb{R}$ such that the following conditions are met:

$$\lambda_i^* \ge 0 \text{ and } \lambda_i^* h_i(x^*) = 0, \text{ for } i = 1, 2, \dots, l \tag{0.1}$$

$$Df(x^*) + \sum_{i=1}^{l} \lambda_i^* Dh_i(x^*) = 0 \tag{0.2}$$

Proof. WLOG assume that $E = \{1, \dots k\}$. For each $i \in \{1, \dots, l\}$, define

$$V_i = \{x \in \mathbb{R}^n : h_i(x) > 0\}, \text{ and } V = \cup_{i=1}^{k} V_i.$$

Then V is open (why?). Let $D^* \subset D$ be such that,

$$D^* = U \cap V \cap \{x \in \mathbb{R}^n : h_i(x) = 0, i = 1, \dots, k\}.$$

Clearly x^* is a local maximum of f in D^* (why?) and rank($Dh_E(x^*) =| E |= k$. Consequently, (why?) there exists a vector $\mu = (\mu_1, \dots, \mu_k)$, $\mu_i \in \mathbb{R}$, such that

$$Df(x^*) + \sum_{i=1}^{k} \mu_i Dh_i(x^*) = 0.$$

For $i = 1, 2, \dots l$, define

$$\lambda_i^* = \begin{cases} \mu_i & i = 1, 2, \dots, k \\ 0 & i = k+1, \dots, l. \end{cases}$$

The proof will be complete if we are able to show that $\mu_i \ge 0, i = 1, 2, \dots, k$ (why ?). We will now show that $\mu_1 \ge 0$, the proof for the rest is similar.

Note that $\exists, y \in \mathbb{R}^n$ such that $Dh_1(x^*)y = 1, Dh_2(x^*)y = 0, Dh_3(x^*)y = 0, \dots, Dh_k(x^*)y = 0$ (why ?). Now for each $i = 1, 2 \dots, k$, define a function $g_i : \mathbb{R}^{k+1} \to \mathbb{R}$ by

$$g_i(\begin{bmatrix} z \\ \eta \end{bmatrix}) = h_i(x^* + \eta y + Dh_E(x^*)^T z) - \eta Dh_i(x^*)y,$$

with $z \in \mathbb{R}^k$ and $\eta \in \mathbb{R}$. Let $g = (g_1, \dots g_k)$, we have $g(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = 0$ and suppose $A = Dg(0,0)$, Then $A_x = Dh_E(x^*)Dh_E(x^*)^T$, is invertible(why ?).

By the Implicit function theorem, we have

$$\exists N \text{ open in } \mathbb{R} : 0 \in N, \text{ and } \xi : N \to \mathbb{R}^k : \xi(0) = 0 \text{ and } g(\xi(\eta), \eta) = 0 \tag{0.3}$$

$$D\xi(0) = 0(\text{ why ?}) \tag{0.4}$$

Define $\tau : N \to \mathbb{R}^n$ such that

$$\tau(\eta) = x^* + \eta y + Dh_E(x^*)\xi(\eta).$$

Note the following properties of τ

(i) $\tau(0) = x^*$.

(ii) $h_1(\tau(\eta)) = \eta$ and $h_i(\tau(\eta)) = 0$ for $i = 2, \dots k$. So $h_i(\tau(\eta)) \geq 0$, for $i = 1, \dots, k$ if $\eta \geq 0$.

(iii) $D(\tau(0)) = y$.

Therefore, (why ?),

$$Df(x^*)y = Df(\tau(0))D(\tau(0)) = \lim_{\eta \downarrow 0} \frac{f(\tau(\eta)) - f(\tau(0))}{\eta} \leq 0. \tag{0.5}$$

Also

$$Df(x^*)y = -(\sum_i = 1^l \lambda_i^* Dh_i(x^*))y = -\mu_1 \tag{0.6}$$

Matching (0.6) and (0.5) we are done. □