Not Due

1. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $g : \Omega \to \mathbb{R}$ be a measurable function. Define $\nu(A) = \mu(g^{-1}(A))$ for any Borel set A. Show that ν is a measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. ν is called the measure on R induced by g.

2. Let $(\Omega, \mathcal{B}, \mu)$ be a measure space and let $B \in \mathcal{B}$. Define a function on $\nu : \mathcal{B} \to [0, \infty]$ such that $\nu(C) = \mu(C \cap B)$, $\forall C \in \mathcal{B}$.

   (a) Show that for any $\mathcal{B}$ measurable function $f : \Omega \to \bar{\mathbb{R}}$,

   $$\int f d\nu = \int f 1_B d\mu,$$

   i.e. if one of them exists then so does the other and they are equal.

3. Let $\{f, f_n : n \geq 1.\}$ are $\bar{\mathbb{R}}$ valued measurable functions on $(\Omega, \mathcal{A}, \mu)$.

   (a) Suppose $f_n \uparrow f$ and $\exists h : \Omega \to \bar{\mathbb{R}}$ with $\int h_- d\mu < \infty$ such that $f_n \geq h$ for all n. Then show that $\int f_n d\mu \uparrow \int f d\mu$

   (b) Provide a counter example if the above hypothesis is violated.

4. Show that for any Borel function $f : \mathbb{R} \to \mathbb{R}$ and any $y \in \mathbb{R}$, $\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} f(x - y)dx$.

5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a measurable function, such that

   (a) For each $x \in \mathbb{R}$, $f(x, \cdot)$ is integrable with respect to Lebesgue measure on $\mathbb{R}$.

   (b) $\forall (x, y) \in \mathbb{R}^2$, $\frac{\partial}{\partial x} f(x, y)$ exists and

   (c) there is an $h : \mathbb{R} \to \mathbb{R}$ integrable such that $\mid \frac{\partial}{\partial x} f(x, y) \mid \leq h(y)$ for all $x, y \in \mathbb{R}$.

   Show that
   $$\frac{d}{dx} \int_{\mathbb{R}} f(x, y)dy = \int_{\mathbb{R}} \frac{\partial}{\partial x} f(x, y)dy$$

6. Show that $\int_1^\infty e^{-s} \log s ds = \lim_{n\to\infty} \int_1^n (1 - \frac{s}{n})^n \log s ds$ and $\int_0^1 e^{-s} \log s ds = \lim_{n\to\infty} \int_0^1 (1 - \frac{s}{n})^n \log s ds$

7. Let X be a random variable on the probability space $(\Omega, \mathcal{B}, P)$, with distribution P_X. Consider the random variable $\tilde{X}$ on the probability space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, P_X)$ defined by $\tilde{X}(x) = x$. Then $P_{\tilde{X}} = P_X$.

8. Let $F : \mathbb{R} \to [0, 1]$ be a distribution function as defined in class. Then show that there is a random variable $X : ((0, 1], \mathcal{B}, \lambda) \to \mathbb{R}$, (where $\mathcal{B}$ is the Borel σ-algebra and λ is Lebesgue measure),such that $P_X = P$

9. Let $\Omega_1 = \Omega_2 = \mathbb{N}$. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{P}(\mathbb{N})$. Let $\mu_i, i = 1, 2$ be two measures on $(\Omega_i, \mathcal{B}_i)$ respectively defined by $\mu_1(\{n\}) = \mu_2(\{n\}) = \frac{1}{2^n}, \forall n \in \mathbb{N}$. Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ such that

   $$f(m, n) = \begin{cases} n2^{2n} & \text{if } m = n,\, n \in \mathbb{N} \\ -n2^{2n+1} & \text{if } m = n - 1,\, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that f is measurable with respect to the product σ-algebra $\mathcal{B}_1 \otimes \mathcal{B}_2$.

   (b) Show that $\int \int f d\mu_1 d\mu_2 \neq \int \int f d\mu_2 d\mu_3$ and discuss why Fubini's Theorem does not apply here.

10. Show that for any non-negative random variable $X \geq 0$, $E(X) = \int_0^\infty P(X \geq x)dx = \int_0^\infty P(X \geq x)dx$.

11. For a measurable function f on $(\Omega, \mathcal{B}, \mu)$ Let $B \in \mathcal{B}$ be such that on $f(B) \subset [a, b]$ where $a, b \in \mathbb{R}$. Show that $\int_B f d\mu = \alpha\mu(B)$ for some $\alpha \in [a, b]$. Moreover $\alpha \in (a, b)$ unless f is μ a.e. constant on B.

12. Consider $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the countable co-countable σ-algebra on $\mathbb{R}$. Show that the diagonal $D = \{(x, x) : x \in \mathbb{R}\}$ is not in the product σ-algebra $\mathcal{B} \otimes \mathcal{B}$ but its sectionals D_x and D^y are in $\mathcal{B}$ for all $x, y \in \mathbb{R}$.

13. Let $(\Omega_i, \mathcal{B}_i), 1 \leq i \leq n$ be measurable spaces. Let $(\Omega, \mathcal{B})$ be the product σ-algebra. For each $1 \leq i \leq n$, let $\pi_i : \Omega \to \Omega_i$ be the coordinate projections. Show that

    (a) $\mathcal{B}$ is the smallest σ-algebra on Ω making all the $\pi_i, 1 \leq i \leq n$ measurable.

    (b) Let $(\hat{\Omega}, \hat{\mathcal{B}}$ be another measurable space. Show that $f : (\hat{\Omega}, \hat{\mathcal{B}} \to (\Omega, \mathcal{B})$ is measurable if and only $\pi_i \circ f : (\hat{\Omega}, \hat{\mathcal{B}} \to (\Omega_i, \mathcal{B}_i)$ is measurable for all $1 \leq i \leq n$.