Due: October 4th, Tuesday

Probems to be turned in are: 1,3,6

1. A function $f : \mathbb{R} \to \mathbb{R}$ is called upper-semicontinuous if $\forall x \in \mathbb{R}$and $\forall \epsilon > 0, \exists \delta \equiv \delta(x, \epsilon) > 0$ such that $y \in (x - \epsilon, x + \epsilon)$ implies $f(y) < f(x) + \epsilon$. Show that upper semi-continuous functions are measurable.

2. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Let $\mathcal{F}$ be a field generating $\mathcal{A}$. Show that if $f : \Omega \to \mathbb{R}$ is any $\mathcal{A}$ measurable function, then for each $\epsilon > 0$, $\exists$ a simple measurable function $g = \sum_{i=1}^{n} x_i 1_{F_i}$ with $F_i \in \mathcal{F}$ such that $\mu(\omega : \mid f - g \mid > \epsilon) < \epsilon$.

3. Let $(\Omega, \mathcal{B})$ and $(E, \mathcal{E})$ be measurable spaces. Let $f : \Omega \to E$ be any function.

   (a) Show that
   $$\sigma(f) = \{f^{-1}(B) : B \in \mathcal{E}\},$$
   is a σ-algebra. This is usually referred to as the σ-algebra generated by f.

   (b) Show that if $\mathcal{S}$ is a collection of subsets of E such that $\sigma(\mathcal{S}) = \mathcal{E}$ then show that $\sigma(f) = \sigma\{f^{-1}(S) : S \in \mathcal{S}\}$.

4. Let $f : \Omega \to \mathbb{R}$ be any function and let $\mathcal{B} = \sigma(f)$. Show that a function $g : \Omega \to \mathbb{R}$ is $\mathcal{B}$ measurable if and only if there is a borel function $h : \mathbb{R} \to \mathbb{R}$ such that $g = h \circ f$.

5. For each of the following functions, describe $\sigma(g)$

   (a) $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = x^2$,

   (b) $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = x(mod(2))$

   (c) $g : \mathbb{R}^2 \to \mathbb{R}$ such that $g(x, y) = \mid x - y \mid$.

   (d) $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = -3x$.

6. Let f be a non-negative measurable function defined on the measure space $(\Omega, \mathcal{B}, \mu)$. Define
   $$\mu_f : \mathcal{B} \to [0, \infty] \text{ by } \mu_f(E) = \int_E f d\mu.$$
   show that: (i) μ_f is a measure defined on B .
   (ii) μ_f is σ-finite if and only f is finite almost everywhere.
   (iii) $E \in \mathcal{B}, \mu(E) = 0 \Rightarrow \mu_f(E) = 0$.