1. **Branching Chain** Let X_0 be a $\{0\} \cup \mathbb{N}$ valued random variable with distribution μ. For $n \geq 1$,

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_i^{(n)}$$

where $\xi_k^{(n)}$ are i.i.d random variables with common distribution given by ξ, such that

$$p_k = P(\xi = k), \text{ for all } k \geq 0.$$

Assume $p_0 + p_1 < 1$.

(a) Show that X_n is a Markov chain on $\{0\} \cup \mathbb{N}$ with initial distribution μ with transition matrix P given by

$$p_{ij} = P(\sum_{k=1}^{i} \xi_k = j),$$

where ξ_k are i.i.d ξ.

(b) Deduce that with probability 1 either $X_n = 0$ for some $n \geq 1$ or $X_n \to \infty$ as $n \to \infty$

(c) Let $m = E[\xi]$. Show that $E[X_n] = m^n$ for all $n \geq 1$.

(d) **Generating Function:** Let $f(s) = \sum_{k=0}^{\infty} p_k s^k$ for $\mid s \mid \leq 1$.

i. Show that $\sum_{j=0}^{\infty} p_{ij} s^j = (f(s))^i$ for all $i \geq 1$ and $\mid s \mid \leq 1$

ii. Consider the iterates of f,

$$f_0(s) = s, f_1(s) = f(s), f_{n+1}(s) = f(f_n(s)).$$

Show that

$$\sum_{j=0}^{\infty} p_{ij}^n s^j = \sum_{j=0}^{\infty} p_{ij}^{n-1} (f(s))^j = (f_n(s))^i, \text{ for all } i \geq 1.$$

(e) **Smallest Root:**

i. Show that f is strictly convex and increasing in $[0, 1]$.

ii. If $m \leq 1$ then $f(t) > t$ for all $t \in [0, 1)$.

iii. If $m > 1$ then $f(t) = t$ has a unique root in $[0, 1)$.

iv. Show that there is a $q \in [0, 1]$ such that q is the smallest solution to $f(q) = q$. Further if $m \leq 1$ then $q = 1$ and if $m > 1$ then $q < 1$.

(f) Show that if

$$\begin{array}{ll} t \in [0,q) & \text{then } f_n(t) \uparrow q \text{ as } n \to \infty \\ t \in [q,1) & \text{then } f_n(t) \downarrow q \text{ as } n \to \infty \\ t = q & \text{then } f_n(t) = t \text{ for all } n \geq 1. \end{array}$$

(g) **Extinction Probability**

$$P(X_n = 0 \text{ for some } n \geq 0) = \begin{cases} 1 & \text{if } m \leq 1 \\ q & \text{if } m < 1, \end{cases}$$

where $q < 1$ is as in (e).

2. **Wright-Fisher Model** In each generation there are m alleles, some of type A and some of type a. The types of alleles in generation $n+1$ are found by choosing randomly (with replacement) from the types in n-th generation. If X_n denotes the number of alleles of type A in generation n, then X_n be a discrete time Markov chain on $\{0, 1, \ldots, m\}$ with transition probability matrix P given by

$$p_{ij} = \binom{m}{j} \left(\frac{i}{m}\right)^j \left(\frac{m-i}{m}\right)^{m-j}.$$

(a) Find the communicating classes of X_n.

(b) Find $h(i) = P(X_n = m \text{ for some } n)$.

3. **Moran Model** Consider the birth-death chain on $\{0, 1, 2 \ldots m\}$ with transition probabilities given by

$$p_{i,i-1} = \frac{i(m-i)}{m^2},\ p_{i,i} = \frac{i^2 + (m-i)^2}{m^2},\ p_{i,i+1} = \frac{i(m-i)}{m^2},$$

when $1 < i < m$ and $p_{0,0} = p_{m,m} = 1$.

(a) Can you give a genetic interpretation for this model as in Wright-Fisher model ?

(b) Find $k(i) = E_i[T]$ where $T = \inf\{k \geq 0 : X_k \in \{0, 1\}\}$