1. Let X_n[1] be a Markov chain on state space S with transition matrix P. The period of $i \in S$ is defined as
$$d(i) = g.c.d.\{m \geq 1 : p_{ii}^m > 0\} \tag{1}$$
If $p_{ii}^m = 0$ for all $m \geq 1$, then $d(i)$ is defined to be 0. Let $i \in S$, then show that $d(i) = 1$ if and only if there is an n_0 such that $p_{ii}^n > 0$ for all $n \geq n_0$.

2. Suppose Pyare Lal and Lajo have two rupees each. They decide to play a game according to the following rules. At each turn a coin will be tossed, if it turns up heads then Pyare Lal will give Lajo one rupee. If it turns up tails then Lajo will give Pyare Lal one rupee. The game ends if any one player runs out of money. Let X_n be the wealth of, say, Lajo at time n. Then X_n is a Markov chain on state space $S = \{0, 1, 2, 3, 4\}$ with $X_0 = 2$ and transition matrix
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$
Show that $F = \inf\{n \geq 1 : X_n \in \{0, 4\}\}$ is a stopping time. Can you find the distribution of X_F ?

3. Let X_n be the simple symmetric walk on with $X_0 = 0$. Let $a > 0$. Let
$$T_a = \inf\{n \geq 1 : X_n = a\}.$$
   (a) Show that T_a is a stopping time and the 'inf' in T_a is actually a minimum with probability one.
   (b) **(Reflection Principle)** Suppose $M_n = \max\limits_{0 \leq i \leq n} X_i$. Show that
$$P(M_n \geq a, X_n < a) = P(M_n \geq a, X_n > a).$$
(Hint: Apply the strong markov property at T_a and symmetry of the distribution of the Bernoulli trials.)

[1] $d(i)$ is said to be the period of i and X_n is said to be aperiodic if $d(i) = 1$ for all $i \in S$.

4. Let X_n be a Markov chain on state space $S = \{0, 1, 2, \ldots, N\}$ with initial distribution μ and transition matrix given by:

$$p_{ij} = \begin{cases} 1 & \text{if } i = 0, j = 0 \text{ or } i = N, j = N \\ \frac{1}{2} & \text{if } i = j + 1, 1 \le i \le N - 1 \\ \frac{1}{2} & \text{if } i = j - 1, 1 \le i \le N - 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) For $i \in S$, let $D = \min\{n \ge 0 : X_n \in \{0, N\}\}$. Find $f : S \to \mathbb{R}$ where $f(i) = E_i(D)$

(b) Let $g : S \to \mathbb{R}$ be defined as $g(i) = P_i(T_N < T_0)$. Find g.

5. Let X_n be a Markov Chain on state space S with transition matrix P. Let $i \in S$. Let

$$T_i^{(0)} = 0, \text{ and } T_i^{(r)} = \inf\{k > T_i^{(r-1)} | X_k = i\}.$$

Show that

(a) $T_i^{(r)}$ are stopping times.

(b) $\{S^{(r)i}\}_{r \ge 1}$ are independent.

(c) $\mathbb{P}(S_i^{(r)} = n \mid T_i^{(r-1)} < \infty) = \mathbb{P}_i(T_i = n)$