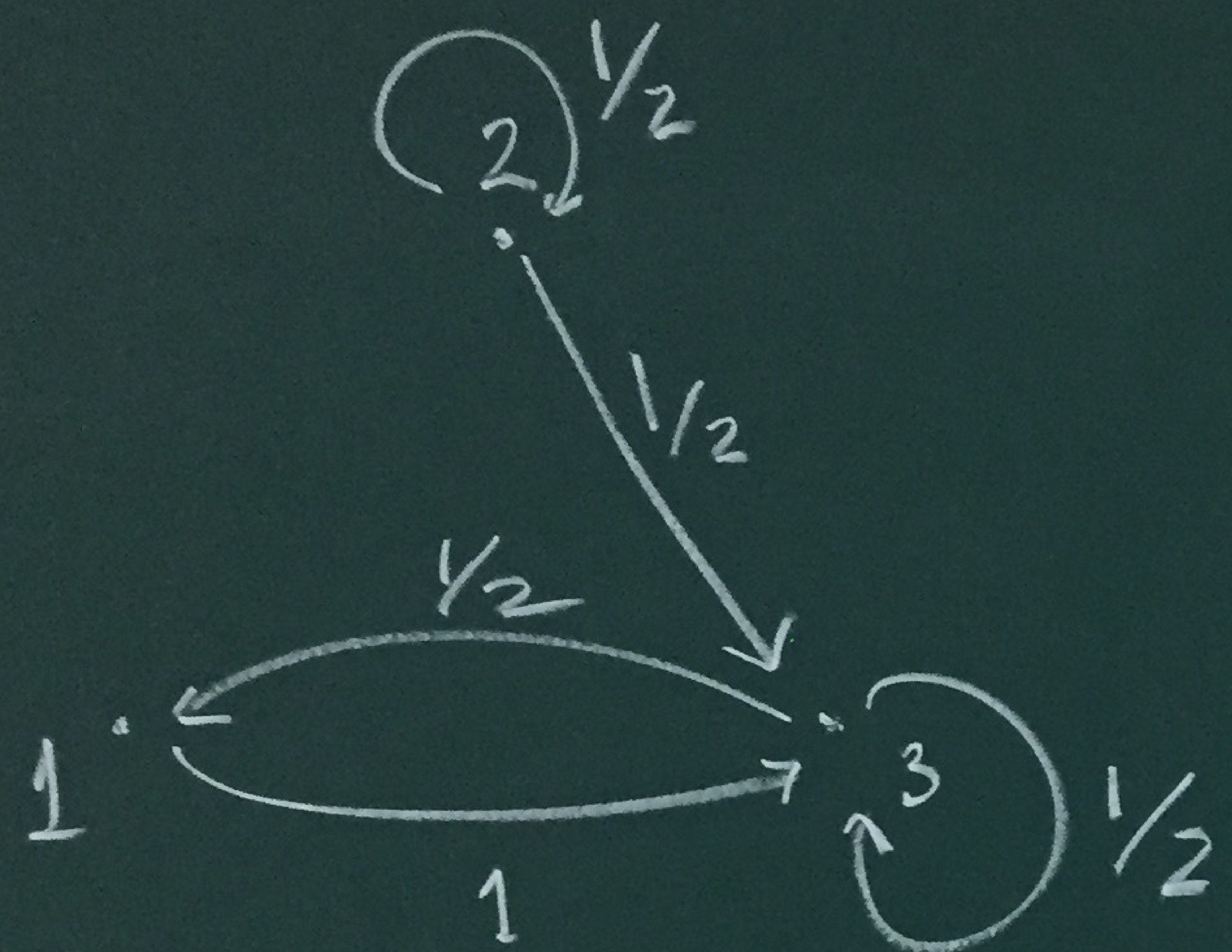
1. $\{X_n\}_{n\geq 0}$ Markov chain. $S=\{1,2,3\}$ $P=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

(a)



(b) WTS that $p_{11}^{n} = \frac{1}{5} + \left(\frac{1}{2}\right)^n \left\{ \frac{4}{5}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{5}\sin\left(\frac{n\pi}{2}\right) \right\}$ for $n \geq 3$.

$P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 0 & 3/4 \end{bmatrix}$; $P^3 = P^2 P = \begin{bmatrix} 1/4 & 0 & 3/4 \\ 1/4 & 1/8 & 5/8 \\ 3/8 & 0 & 5/8 \end{bmatrix}$

$$P^4 = P^3 \cdot P = \begin{bmatrix} 3/8 & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

We see that $p_{11}^{4} = 3/8$.

whereas from the expression we get

$$1/5 + \frac{1}{2^4}\left\{\frac{4}{5}\cos(2\pi) - \frac{2}{5}\sin(2\pi)\right\} = \frac{1}{5} + \frac{4}{16 \times 5} = 1/4$$

Since $3/8 \neq 1/4$, the expression is incorrect. $\blacksquare$

Q2)

$X \sim \text{Uniform}\{1,2,3,4,5,6\}$

a) Assuming $P(\text{heads}) = p$

$$P(Y=y \mid X=n) = \begin{cases} \binom{n}{y} p^y (1-p)^{n-y} & \text{if } 0 \le y \le n \le 6 \quad y, n \in \mathbb{N} \\ 0 & \text{o.w} \end{cases}$$

$\therefore Y_{\mid X=n} \sim \text{Bionomial}(n, p)$

b) $P(X=x, Y=y) = P(Y=y \mid X=x)\, P(X=x)$

if $x \in \{1,2,3,4,5,6\}$ $0 \le y \le x$, $y, x \in \mathbb{N}$ then

$$= \begin{cases} \binom{x}{y} p^y (1-p)^{x-y} \times \frac{1}{6} & 0 \le y \le x \le 6,\ y, x \in \mathbb{N}, x \ge 1 \\ 0 & \text{o.w} \end{cases}$$

c) $$P(Y=y) = \sum_{x=y}^{6} P(X=x, Y=y) = \binom{y}{y} p^y (1-p)^{y-y} + \binom{y+1}{y} p^y (1-p)^{(y+1)-(y)} + \cdots + \binom{6}{y} p^y (1-p)^{6-y}$$

$$= \begin{cases} p^y \sum_{i=0}^{6-y} \binom{y+i}{i} (1-p)^i & \text{if } 0 \le y \le 6,\ x > 0 \\ 0 & \text{o.w} \end{cases}$$

(d) $P(X=x \mid Y=6)$

$x \geq 6$ (if 6 times head occurs $\Rightarrow$ atleast coin was tossed 6 times)

But $x \in \{1, 2, \ldots, 6\}$

$\Rightarrow \quad x = 6$

$$P(X=x \mid Y=6) = \begin{cases} 1 & x=6 \\ 0 & \text{o w} \end{cases}$$

mes)

(e) $P(X=x|Y=0)$

$$= \frac{P(X=x, Y=0)}{P(Y=0)}$$

$$P(X=x, Y=0) = {}^{x}C_0 \, p^0 \cdot (1-p)^{x} \tfrac{1}{6} = \frac{(1-p)^x}{6}$$

$$P(Y=0) = \sum_{x=1}^{6} {}^{x}C_0 (1-p)^x \frac{1}{6} \cdot p^0 = \left(\frac{1-(1-p)^6}{p}\right)(1-p)$$

$$\Rightarrow P(X=x|Y=0) = \begin{cases} \dfrac{p(1-p)^{x-1}}{1-(1-p)^6} & 0 < x \leq 6 \\ 0 & \text{O.W} \end{cases}$$

(f)

$X|_{Y=6} \sim \text{constant}\{6\}$

i.e. $P(X=x|Y=6) = \begin{cases} 1 & x=6 \\ 0 & \text{o.w} \end{cases}$

But $X \sim \text{Uniform}\{1, 2, \ldots, 6\}$

if X & Y were independent, then distribution of

$X|_{Y=6}$ & X should be same.

$\therefore$ X & Y are not independent

3 $X, Y \sim \text{Uniform}\{1, \ldots, n\}$, independent

$Z = \text{Max}(X, Y)$ $\quad W = \text{Min}(X, Y)$

$\text{Range}(Z) = \text{Range}(W) = \{1, \ldots, n\} =: S$

$Z \geq W$ w.p 1 as $Z = \text{Ma}(X, Y)$ & $W = \text{Min}(X, Y)$

for $k, l \in S$ $\mathbb{P}(Z = k, W = l) = 0$ if $k < l$.

if $k \geq l$ $\mathbb{P}(Z = k, Y = l) = \begin{cases} \mathbb{P}(X = k, Y = k) & \text{if } k = l \\ \mathbb{P}(X = k, Y = l) + \mathbb{P}(X = l, Y = k) & \text{if } k > l \end{cases}$

$\underset{\text{independence}}{=} \begin{cases} \frac{1}{n^2} & \text{if } k = l \\ \frac{2}{n^2} & \text{if } k \neq l. \end{cases}$

b)

b) for $k \in S$, $E[Z \mid W=k] = \sum_{l \in S} l\, \mathbb{P}(Z=l, W=k)$

$(\mathbb{P}(W=k) > 0)$

But $\mathbb{P}(Z=l, W=k) = 0$ if $l < k$

$$\therefore E[Z \mid W=k] = \sum_{l=k}^{n} l\, \mathbb{P}(Z=l \mid W=k) = k\, \mathbb{P}(Z=k \mid W=k) + \sum_{l=k+1}^{n} l\, \mathbb{P}(Z=l \mid W=k)$$

$$= k \frac{\mathbb{P}(Z=k, W=k)}{\mathbb{P}(W=k)} + \sum_{l=k+1}^{n} l \frac{\mathbb{P}(Z=l, W=k)}{\mathbb{P}(W=k)}$$

($\sum$ is 0 if $k=n$)

$$= \frac{1}{\mathbb{P}(W=k)} \left(k \frac{1}{n^2} + \frac{2}{n^2}\left(\frac{n(n+1)}{2} - \frac{k(k+1)}{2}\right) \right)$$

$$= \left(\frac{k}{n^2} + \frac{(n-k)(n+k+1)}{n^2} \right) \frac{1}{\mathbb{P}(W=k)}$$

$$\mathbb{P}(W=k) = \sum_{l=k}^{n} \mathbb{P}(Z=l, W=k) = \frac{1}{n^2} + \frac{2}{n^2}(n-k)$$

$$\mathbb{E}\left[Z \,\middle|\, W=k\right] = \frac{\dfrac{k+(n-k)(n+k+1)}{n^2}}{\dfrac{1+2(n-k)}{n^2}} = \frac{n^2-k^2+n}{1+2(n-k)}$$