

Recall

• $(X_n)_{n \geq 0}$ - Markov chain on S , transition matrix P
and initial distribution

- π a probability on S is a stationary distribution if $\sum_i \pi(i) p_{ij} = \pi(j)$

$\lim_{n \rightarrow \infty} p_{ij}^{(n)} = \pi(j) \quad \forall j \in S \Rightarrow \pi(\cdot)$ is stationary

Positive recurrent $\left\{ \begin{array}{l} \text{irreducible \& aperiodic} \\ \text{and has a} \\ \text{stationary distribution} \end{array} \right. \Rightarrow \pi(i) = \frac{1}{E_i[T_i]}$

$$\Rightarrow \lim_{n \rightarrow \infty} p_{ij}^{(n)} = \pi(j)$$

$\frac{N_n(i)}{n}$	proportion
$\rightarrow \frac{1}{E_i[T_i]}$	$n \rightarrow \infty$

Theorem 1.

Special

Remark:

Theorem 1.9.15

let $(X_n)_{n \geq 0}$ be irreducible
 $f: S \rightarrow \mathbb{R}$ bounded function

Suppose $(X_n)_{n \geq 0}$ is positive recurrent
 Then

$$\frac{1}{n} \sum_{k=1}^n f(X_k) \longrightarrow \sum_{s \in S} \pi(s) f(s) \quad \text{with probability one}$$

where $\pi(s)$ is the unique stationary distribution
 given by $\pi(i) = \frac{1}{E_i[T_i]}$

Special case: $f(s) = \begin{cases} 1 & s=i \\ 0 & \text{otherwise} \end{cases}$

Thm 1.9.8

Thm 1.9.10

$$\pi(i) = \frac{1}{E_i[T_i]}$$

$$\frac{\sum_{k=1}^n f(X_k)}{n} = \frac{N_n(i)}{n} \xrightarrow[n \rightarrow \infty]{\text{w.p.1}} \frac{1}{E_i[T_i]} = \sum_{s \in S} \pi(s) f(s)$$

Remark: $E_\pi[f] = \sum_{s \in S} \pi(s) f(s)$

Proof: $\left| \frac{1}{n} \sum_{k=1}^n f(X_k) - \sum_{s \in S} \pi(s) f(s) \right|$

$$= \left| \frac{1}{n} \sum_{k=1}^n \sum_{i \in S} f(i) \mathbb{1}_{\{X_k=i\}} - \sum_{i \in S} f(i) \pi(i) \right|$$

$\sum_{k=1}^n f(X_k) = \sum_{i \in S} f(i) N_n(i)$
 $N_n(i) = \sum_{k=1}^n \mathbb{1}_{\{X_k=i\}}$

$$= \left| \sum_{i \in S} f(i) \left(\frac{N_n(i)}{n} - \pi(i) \right) \right|$$

$$\leq \sum_{i \in S} |f(i)| \left(\frac{N_n(i)}{n} - \pi(i) \right)$$

let $J \subseteq S$
 $|J| < \infty$

$$\leq M \left[\sum_{i \in J} \frac{N_n(i)}{n} - \sum_{i \in J} \pi(i) \right]$$

Proof:-

$$\left| \frac{1}{n} \sum_{k=1}^n f(X_k) - \sum_{i \in S} \pi(i) f(i) \right|$$

$$= \left| \frac{1}{n} \sum_{k=1}^n \sum_{i \in S} f(i) \mathbb{1}(X_k = i) - \sum_{i \in S} \pi(i) f(i) \right|$$

$\sum_{k=1}^n f(X_k)$ is finite sum
 $\frac{1}{n}$ sum

$$N_n(i) = \sum_{k=1}^n \mathbb{1}(X_k = i)$$

$$\left| \sum_{i \in S} f(i) \frac{N_n(i)}{n} - \sum_{i \in S} \pi(i) f(i) \right|$$

$$= \left| \sum_{i \in S} f(i) \left(\frac{N_n(i)}{n} - \pi(i) \right) \right|$$

$$\leq \sum_{i \in S} |f(i)| \left(\frac{N_n(i)}{n} - \pi(i) \right)$$

let

$$J \subseteq S$$

$$|J| < \infty$$

$$\leq M \left[\sum_{i \in J} \left| \frac{N_n(i)}{n} - \pi(i) \right| + \sum_{i \notin J} \left(\frac{N_n(i)}{n} + \pi(i) \right) \right]$$

$$\exists M > 0 \quad |f(s)| \leq M \quad \forall s \in S$$

$$\text{Note:- } \left| \sum_{s \in S} \pi(s) f(s) \right|$$

$$\leq M \sum_{s \in S} \pi(s) = M$$

$$\begin{aligned} & \sum_{i \in J} \frac{N_n(i)}{n} + \pi(i) \\ &= \sum_{i \in J} \left(\frac{N_n(i)}{n} - \pi(i) \right) + 2 \sum_{i \in J} \pi(i) \\ &= \sum_{i \in J} \left(\frac{N_n(i)}{n} - \pi(i) \right) - \sum_{i \in J} \left(\frac{N_n(i)}{n} - \pi(i) \right) + 2 \sum_{i \in J} \pi(i) \\ &\leq 0 + \sum_{i \in J} \left| \frac{N_n(i)}{n} - \pi(i) \right| + 2 \sum_{i \in J} \pi(i) \end{aligned}$$

$$\leq M \left[2 \sum_{i \in J} \left| \frac{N_n(i)}{n} - \pi(i) \right| + 2 \sum_{i \in J} \pi(i) \right]$$

Let $\varepsilon > 0$ be given

$$\exists J \subseteq S \quad |J| < \infty$$

$$\text{such that } \sum_{i \in J} \pi(i) < \varepsilon$$

w.p. 1

$$\exists N$$

$$n \geq N$$

$$\left| \frac{N_n(i)}{n} - \pi(i) \right| < \varepsilon \quad \forall n \geq N$$

$$\forall i \in J$$

$\Rightarrow$ w.p. 1 $\exists N$ such that

$$\left| \frac{1}{n} \sum_{k=1}^n f(X_k) - \sum_{s \in S} \pi(s) f(s) \right| \leq M[2\varepsilon|J| + \varepsilon]$$

$$\text{w.p. 1} \quad \frac{1}{n} \sum_{k=1}^n f(X_k) \xrightarrow{a.s.} \sum_{s \in S} \pi(s) f(s) \quad \text{as } n \rightarrow \infty$$

$$= \varepsilon[2M|J| + M]$$

1.10 Reversing Time

Markov chain — Part
 $X_0 \stackrel{d}{=} \mu$

Present

Future are independent given the present

- observe that there may be symmetry with respect to time.
- $X_0 \stackrel{d}{=} \mu$ & $(X_n)_{n \geq 0}$ a periodic & irreducible, $(X_n)_{n \geq 0}$ converges in distribution to stationary distribution π . So to preserve symmetry we would $\mu = \pi$.

Thm 1.10.1

let $(X_n)_{n \geq 0}$ be a Markov chain on S with ^{IRREDUCIBLE} transition matrix P

& stationary distribution π . let $X_0 \stackrel{d}{=} \pi$

$N \geq 1$

consider $(X_n)_{0 \leq n \leq N}$

let $Y_n = X_{N-n}$ $0 \leq n \leq N$

Then $(Y_n)_{0 \leq n \leq N}$ is a Markov chain on S with $Y_0 \stackrel{d}{=} \pi$

& transition matrix $\tilde{P}$ which is irreducible given by

$$\tilde{P}_{ij} = \frac{P_{ji} \pi(j)}{\pi(i)}$$

$\forall i, j \in S.$

AND

$$\pi \tilde{P} = \pi$$

Proof $X_0 \stackrel{d}{=} \pi$, π -stationary $\Rightarrow X_N \stackrel{d}{=} \pi \Rightarrow Y_0 \stackrel{d}{=} \pi$

$(X_n)_{n \geq 0}$ is irreducible & has stationary distribution π

$\Rightarrow (X_n)_{n \geq 0}$ is positive recurrent
Thm 1.9.10 & $\pi(i) = \frac{1}{E_i(T_i)} > 0 \quad \forall i \in S$

$\Rightarrow \hat{P}_{ij}$ is well defined for $\forall i, j \in S$

clearly, $\hat{P}_{ij} \geq 0$ & $\sum_{j \in S} \hat{P}_{ij} = \sum_{j \in S} \frac{p_{ij} \pi(j)}{\pi(i)} \stackrel{\pi R P}{=} \frac{\pi(i)}{\pi(i)} = 1$

Let $i_0, i_1, \dots, i_N \in S$

$P(Y_0 = i_0, \dots, Y_N = i_N)$

By definition of $(X_n)_{0 \leq n \leq N}$

iterate inductively

$\Rightarrow (Y_n)_{0 \leq n \leq N}$

$$\Rightarrow Y_0 \stackrel{d}{=} \pi$$

π

N

$$i \in S$$

$$= 1$$

Let $i_0, i_1, \dots, i_N \in S$

$$\mathbb{P}(Y_0 = i_0, \dots, Y_N = i_N) = \mathbb{P}(X_0 = i_N, X_1 = i_{N-1}, \dots, X_N = i_0)$$

By definition
of $(X_n)_{0 \leq n \leq N}$

$$= \pi(i_N) p_{i_N i_{N-1}} \dots p_{i_2 i_1} p_{i_1 i_0}$$

$$= \pi(i_N) p_{i_N i_{N-1}} \dots p_{i_2 i_1} \tilde{p}_{i_0 i_1} \frac{\pi(i_0)}{\pi(i_1)}$$

Iterate
inductively

$$= \tilde{p}_{i_{N-1} i_N} \tilde{p}_{i_{N-2} i_{N-1}} \dots \tilde{p}_{i_0 i_1} \pi(i_0)$$

$\Rightarrow (Y_n)_{0 \leq n \leq N}$ is a Markov chain on S with transition matrix $\tilde{P}$,
with $Y_0 \stackrel{d}{=} \pi$

$i \in S \quad j \in S \quad \nexists n_0$ such that

$\Rightarrow \nexists i_1, i_2, \dots, i_{n_0}$

$\Rightarrow \nexists i_{n_0-1}, \dots, i_1$

$\Rightarrow \tilde{p}_{j i}^{n_0} = 0$

$\pi \tilde{P} = \pi$, because

$i \in S, j \in S \quad \nexists n_0$ such that $p_{ij}^{n_0} > 0$

$\Rightarrow \nexists i_1, i_2, \dots, i_{n_0-1} \in S$ such that $p_{i_1 i_2} \dots p_{i_{n_0-1} j} > 0$

$\Rightarrow \nexists i_{n_0-1}, \dots, i_1, 0 \leq i_1 < i_2 < \dots < i_{n_0-1} < i_{n_0} = j$ such that $\tilde{p}_{j, i_{n_0-1}} \dots \tilde{p}_{i_1, i_1} > 0$

$\Rightarrow \tilde{p}_{j, i}^{n_0} > 0 \Rightarrow j \rightarrow i$ for $\forall i$

$\pi \tilde{p} = \pi$, because $\sum_{i \in S} \pi(i) \tilde{p}_{ij} = \sum_{i \in S} \pi(i) \frac{p_{ji}}{\pi(j)} \pi(j) = \pi(j) \sum_{i \in S} p_{ji} = \pi(j)$
 $\square$

Definition 1.10.2:

A stochastic matrix $P \in \Pi$ is
in Detailed balance if

$$\pi(i) p_{ji} = \pi(j) p_{ij} \quad \forall i, j \in S$$

$(X_n)_{n \geq 0}$ be a Markov chain on S with
transition matrix P and initial distribution μ

Remark

For

$Y_0 = \pi$
on S

$(X_n)_{n \geq 0}$

is said to be reversible w.r.t μ if

$\forall N \geq 1$

$(X_{N-n})_{0 \leq n \leq N}$

is also a Markov chain on S

with transition matrix P and initial distribution μ .

Remark. π & P are in detail balance, i.e. $\pi(i) p_{ij} = \pi(j) p_{ji} \forall i, j \in S$.

$$\text{For } j \in S, \sum_{i \in S} \pi(i) p_{ij} = \sum_{i \in S} \pi(j) p_{ji} = \pi(j) \sum_{i \in S} p_{ji} = \pi(j)$$

$\Rightarrow \pi$ is stationary for P

Theorem

(a)

(b)

Proof.

Theorem 1.10.3. $(X_n)_{n \geq 0}$ be a Markov chain on S with irreducible P
 transition matrix P & initial distribution μ

The following are equivalent:

- (a) $(X_n)_{n \geq 0}$ is reversible w.r.t μ
- (b) μ and P are in detailed balance

Proof: (a) $\Rightarrow$ (b) $X_0 \stackrel{d}{=} \mu \wedge X_N \stackrel{d}{=} \mu \quad \forall N \geq 1 \Rightarrow P(X_1 = j) = \mu(j)$
 $\Rightarrow \mu(j) = P(X_1 = j) = \sum_{i \in S} \mu(i) p_{ij} \Rightarrow \mu$ is stationary for $(X_n)_{n \geq 0}$

By theorem 1.10.1 $(X_{N-n})_{0 \leq n \leq N}$

with transition matrix $\tilde{P}$, where

$$\tilde{p}_{ij} = \frac{p_{ji} \mu(j)}{\mu(i)} \quad \forall i, j \in S$$

But $(X_n)_{n \geq 0}$ reversible w.r.t $\mu \Rightarrow$

$$\tilde{P} = P \Rightarrow p_{ij} = \frac{p_{ji} \mu(j)}{\mu(i)} \quad \forall i, j \in S$$

(b) $\Rightarrow$

Thm

$\Rightarrow \mu(i) p_{ji} = \mu(j) p_{ij} \Leftrightarrow \mu \in P$ are in detailed balance.
So (b) holds.

(b) $\Rightarrow \mu$ is stationary for P

Thm 1.10.1 $\Rightarrow \forall n \geq 1$ $(X_{k-n})_{0 \leq k \leq n}$ is M.C on S with transition matrix $\bar{P}$ and $X_n \stackrel{d}{=} \mu$

But $\mu \in P$ are in detailed balance $\Rightarrow \bar{P} = P$

$\Rightarrow (X_n)_{n \geq 0}$ is reversible $\square$

Example

$(X_n)_{n \geq 0}$

is reversible

$\left\{ \begin{array}{l} \mu_i = \sum_{j \in S} \mu_{ij} \\ \mu_{ij} = \mu_{ji} \end{array} \right.$

Example 1.10.4

$$P = (V, E)$$

let μ be weights on P

$$(X_n)_{n \geq 0}$$

be a Markov chain on V with transition matrix P

given by

$$p_{ij} = \frac{\mu_{ij}}{\mu_i}$$

$$\mu_i = \sum_{j \in V} \mu_{ij}$$

$$\mu_{ij} = \mu_{ji} > 0 \iff (i, j) \in E \\ = 0 \text{ otherwise}$$

$$(*) \mu P = \mu$$

Because

$$\sum_{j \in V} \mu_i p_{ij} = \sum_{j \in V} \mu_i \frac{\mu_{ij}}{\mu_i}$$

$$= \sum_{j \in V} \mu_{ji} = \mu_j$$

$$(**) \mu_i p_{ij} = \mu_{ij} = \mu_{ji} = \mu_j p_{ji}$$

Example 1.10.4

$$\Gamma = (V, E)$$

let μ be weights on Γ
(connected graph)

$$(X_n)_{n \geq 0}$$

be a Markov chain on V with transition matrix P

given by

$$P_{ij} = \frac{\mu_{ij}}{\mu_i}$$

$$\mu_i = \sum_{j \in V} \mu_{ij}$$

$$\mu_{ij} = \mu_{ji} > 0 \iff i, j \in E$$

$$= 0 \text{ otherwise}$$

$$(*) \mu P = \mu$$

Because $\sum_{j \in V} \mu_i P_{ij} = \sum_{j \in V} \mu_i \frac{\mu_{ij}}{\mu_i}$

$$= \sum_{j \in V} \mu_{ji} = \mu_i$$

$$(**) \mu_i P_{ij} = \mu_{ij} = \mu_{ji} = \mu_j P_{ji}$$

Suppose $\sum_{i \in V} \mu_i = c < \infty$

then $\pi(i) = \frac{\mu_i}{c}$ is detailed

balance with P by $\otimes \otimes$

$(X_n)_{n \in \mathbb{N}}$ is reversible with respect to $\pi \equiv \frac{\mu}{c}$

$\textcircled{b} \Rightarrow$

Suppose $\sum_{i \in V} \mu_i = c < \infty$

then $\pi(i) = \frac{\mu_i}{c}$ in detailed

balance with P by (xx)

$(X_n)_{n \geq 0}$ is reversible with respect to $\pi \equiv \frac{\mu}{c}$

Example 1.10

$$X_n \equiv$$

$$X_0 = 1$$

At time 1

$$X_1 =$$

At time 2,

Example 1.10.5 (Galton-Watson chain)

$X_n = \#$ of ^(fruit-fly) bacteria alive at time n

$$X_0 = 1$$

At time 1, fly dies and gives birth to ξ_1 many flies

$$X_1 = \xi_1$$

At time 2, all ξ_1 flies die and each give birth to $\left\{ \xi_k^2 \right\}_{1 \leq k \leq \xi_1}$ flies

$$X_2 = \sum_{k=1}^{X_1} \xi_k^2$$

$$\left\{ \xi_k^n \right\}_{n,k \geq 0}$$

$$X_n = \sum_{k=1}^{X_{n-1}} \xi_k^n$$

Trivial case:

Assume $p_1 < 1$

$$\left\{ \sum_k^n \right\}_{n,k \geq 0}$$

l.c.d

$$\sum_k^n \equiv \left(\sum_k^n \right)$$

$$P(\xi = k) = p_k, \quad k \in \{0, 1, \dots, N\}$$

$$\sum_{k=0}^N p_k = 1$$

$$X_n = \sum_{k=1}^{X_{n-1}} \sum_k^n$$

Ex. $(X_n)_{n \geq 0}$ is a Markov chain in $\{0, 1, \dots, N\}$

with absorbing state 0.

Trivial case: $p_1 = 1, p_0 = 1$

Assume $p_1 < 1, p_0 < 1$

Suppose $P(X_{n-1} = i) > 0$, $P(X_n = 0 | X_{n-1} = i) = (P(\xi = 0)) = p_0^i$

If $p_0 > 0$ then $i \rightarrow 0$ & $0 \rightarrow i$ $\Rightarrow$ $i \geq 1$ are transient

$$p_0 = 0 \text{ \& } p_i < 1 \Rightarrow P(\xi \geq 2) > 0$$

$$i \geq 1 \Rightarrow \exists j \in S \quad j > i \quad \& \quad i \rightarrow j$$

$$\text{but } j \nrightarrow i \quad (p_0 = 0)$$

$$\Rightarrow i \geq 1 \text{ transient}$$

$i=0$ - absorbing state

$X_0 = \underline{1}$ $\rightarrow$ $X_n = 0$ for some $n \geq 1$
 $\rightarrow$ $X_n \rightarrow \infty$ as $n \rightarrow \infty$ with positive probability