Summary – Continuous time Markov chains

· Right continuous Process

- $\forall \omega \in \Omega,\ t \geq 0 \quad \exists \varepsilon > 0 \quad X_s = X_t \quad$ for $t < s < t+\varepsilon$

- $\left\{ \mathbb{P}(X_{t_0} = i_0, \dots X_{t_n} = i_n) \right\}_{n \geq 1,\ \{t_i\}_{i=0}^{n}\ \{i_k\}_{k=0}^{n}}$ (finite dimensional distribution)

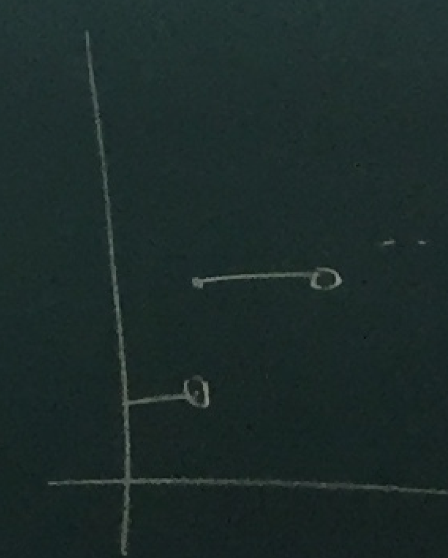
  - characterize the law of $(X_t)_{t \geq 0}$

· $\mathbb{P}(X_t = i \text{ for some } t \geq 0) = 1 - \lim_{n \to \infty} \sum_{\substack{i_1 \dots i_n \neq i \\ a_1 \dots a_n \in \mathbb{Q}}} \mathbb{P}(X_{a_1} = i_1, \dots X_{a_n} = i_n)$

Every right continuous can then be characterized by

← $T_0, \dots$

$S_1, \dots$



[Exp

[Ju

$\mathbb{P}(X_t =$

Example of a

$\{X_t\}_{t \geq 0}$ to be a ...

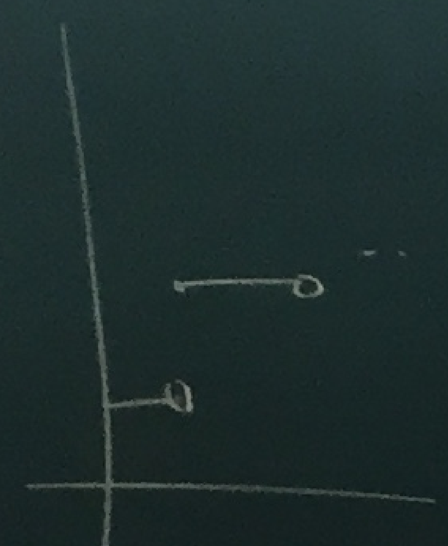
$t+\varepsilon$

...ution)

$X_{a_n} = i_n$)

- $J_0, \dots, J_n$ = $\boxed{\text{Jump times}}$ of $(X_n)_{n \geq 0}$ $\qquad J_0 = 0, \quad J_{n+1} = \inf\{t > J_n \mid X_t \neq X_{J_n}\}$

$S_1, \dots, S_n$ — $\boxed{\text{holding times}}$ $\qquad S_n = \begin{cases} J_n - J_{n-1} & \text{if } J_n < \infty \\ \infty & \text{otherwise} \end{cases}$



[Explosion time] $\qquad \tau = \sup_n J_n = \sum_{n \geq 1}^{\infty} S_n \longrightarrow \begin{cases} \tau \equiv \infty \\ \tau < \infty & \text{— minimal right continuous process by setting } X_t = \infty,\ t \geq \tau \end{cases}$

[Jump chain] $\qquad Y_n = X_{J_n}$

$$\mathbb{P}(X_t = i) = \sum_{n=0}^{\infty} \mathbb{P}(Y_n = i, J_n \leq t < J_{n+1})$$

Example of a continuous time Markov chain on $\{0\} \cup \mathbb{N}$ - Poisson Process of rate λ, $J_0 = 0$

$\{X_t\}_{t \geq 0}$ to be a right continuous process with: $\begin{cases} \{S_k\}_{k \geq 1} \sim \text{iid Exponential}(\lambda), & J_n = S_1 + \dots + S_n \\ & X_t = n \quad J_n \leq t < J_{n+1} \end{cases}$

"Markov" (Property)

EQUIVALEN...

(a) $(X_t)_{t \geq 0}$

($X_t = Y_n$ if $J_n \leq t <$ ...

(b) $\{X_t\}_{t \geq 0}$ & ...

$\mathbb{P}(X_{t+h}$ ...

$\mathbb{P}(X_{t+}$ ...

(c) $(X_t)_{t \geq}$ ...

$X_t \overset{d}{=}$ Poisson ...

$\ne X_{J_n}\}$

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ting $X_t = \infty$
$t \geq \tau$

$+ S_n$
$J_n \leq t < J_{n+1}$

"Markov Property" $(X_{s+t} - X_s)_{t\geq 0}$ is Poisson Process of rate λ & is independent of $(X_r : r \leq s)$

EQUIVALENT DEFINITIONS

(a) $(X_t)_{t\geq 0}$ — $\{S_k\}_{k\geq 1}$ i.i.d exponential (λ)

$(X_t = Y_n$ if $J_n \leq t < J_{n+1})$ $\quad Y_n = n \quad \forall n \geq 1$

$J_n = S_1 + \dots + S_n$

(b) $\{X_t\}_{t\geq 0}$ has independent increments & as $h \downarrow 0$ uniformly in t

$$\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h)$$

$$\mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h)$$

(c) $(X_t)_{t\geq 0}$ stationary & independent increments

$X_t \overset{d}{=} \text{Poisson}(\lambda t)$ $\quad \mathbb{P}(X_{t_k} = i_k \mid X_0 = i_0, \dots X_{t_{k-1}} = i_{k-1}) = p_{i_{k-1} i_k}(t_k - t_{k-1})$

$j \geq i$

$$p_{ij}(t) = \mathbb{P}_i(X_t = j) \equiv p_{0,j-i}(t)$$

$\mathbb{P}(J_0 \leq J_1 \dots \leq J_{j-1} \leq t)$

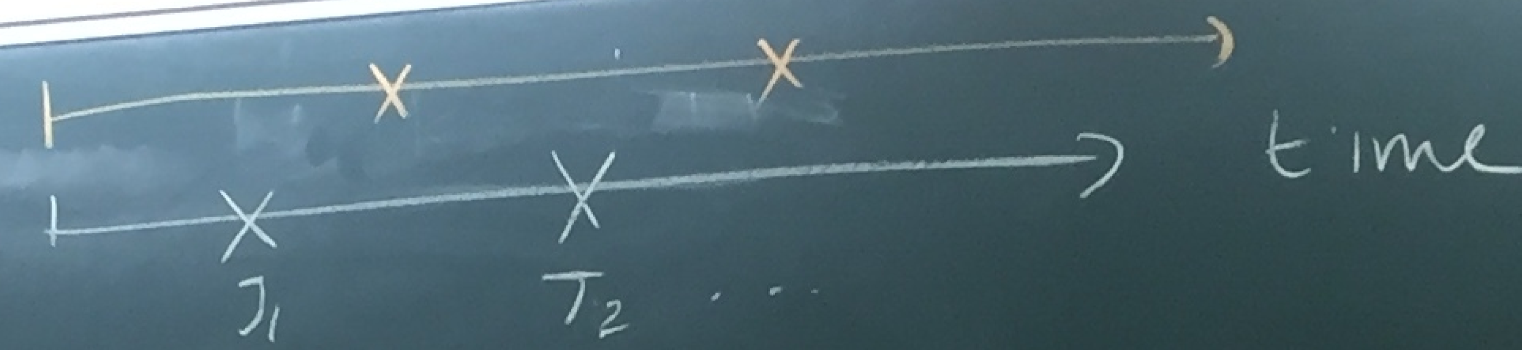
Proof (b) $\Rightarrow$ (c)

$$\begin{cases} p_{i0}'(t) = -\lambda p_{i0}(t) & , p_{i0}(0) = \delta_{i0} \\ p_{ij}'(t) = \lambda p_{i,j-1}(t) - \lambda p_{ij}(t) & , p_{ij}(0) = \delta_{ij} \end{cases}$$

ie $\quad P'(t) = P(t)Q \quad , \quad P(0) = I$

$$\boxed{p_{ij}(t) = e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!}} \qquad Q = \begin{bmatrix} -\lambda & \lambda & & \\ & -\lambda & \lambda & \\ & & \ddots & \ddots \end{bmatrix}$$

Facts - Poisson Process



- $\{X_t\}_{t\geq 0}$ Poisson process with rate μ

  $\{Z_t\}_{t\geq 0}$ Poisson process with rate λ

  $\Bigg\} \{X_t + Z_t\}_{t\geq 0} \to$ Poisson process with rate $\lambda+\mu$

- Condition on one jump in $[s, s+t] \Rightarrow$ time of jump is uniform in $[s, s+t]$.

- Condition on $X_t = n$

$$T_1 \ldots T_n \mid X_t = n \sim f(t_1 \ldots t_n) = \frac{n!}{t^n} \mathbb{1}_{\{0<t_1<\ldots<t_n\}}$$

Exa

$\{X_t\}_{t\geq}$

$\{ S_1$

Example 2 of a continuous time Markov chain on $\{0\} \cup \mathbb{N}$ — Birth Process $\{q_i\}_{i=0}^{\infty}$, $0 \le q_i < \infty$

$\{X_t\}_{t \geq 0}$ — minimal right continuous process taking values in $S = \{0\} \cup \mathbb{N} \cup \{\infty\}$

$$\begin{cases} S_1, S_2, \dots \mid X_0 = i \sim \text{independent exponential}(q_i),\ \text{exponential}(q_{i+1}) \dots \text{random variables} \\ Y_n = i + n \end{cases}$$

Example of "q_i": $q_i = i\lambda$

$E[X_t] = e^{\lambda t}$

Think of a system of particles that give birth at an exponential time with rate λ

$T_1 \dots T_i \sim \exp(\lambda) \Rightarrow \min(T_1 \dots T_i) \sim \exp(i\lambda)$

— $\sum_{j=0}^{\infty} \frac{1}{q_j} < \infty \Rightarrow \tau < \infty$ w.p. 1

— $\sum_{j=0}^{\infty} \frac{1}{q_j} = \infty \Rightarrow \tau = \infty$ w.p. 1

$\}_{i=0}^{\infty} \quad 0 \le q_i < \infty$

variables

exponential

$\ldots T_i) \sim \exp(\lambda)$

"Markov Property" $(X_{t+s})_{t\ge 0}$ given $X_s = i$ is a birth process with rates $q_i, q_{i+1}, \ldots$ & is independent of $(X_r : r \le s)$

EQUIVALENT DEFINITIONS

[Jump-chain + Holding time]

(a) $(X_t)_{t\ge 0}$ — $\{S_k\}_{k\ge 1} \mid X_0 = i$ i.i.d. $\{\text{Exponential}(q_{i+k-1})\}_{k\ge 1}$

$(X_t = Y_n$ if $T_n \le t < T_{n+1})$ $\quad Y_n = n+i \quad \forall n \ge 1$

$J_n = S_1 + \ldots + S_n$

[infinitesimal definition]

(b) $\{X_t\}_{t\ge 0}$ has independent increments & on $h \to$ uniformly in t

$$\mathbb{P}(X_{t+h} = i \mid X_t = i) = 1 - q_i h + o(h)$$

$$\mathbb{P}(X_{t+h} = i+1 \mid X_t = i) = q_i h + o(h)$$

[transition definition]

(c) $(X_t)_{t\ge 0}$ for all $k \ge 0$, $i_0, \ldots i_k$ & $t_0, \ldots t_k$

$$\mathbb{P}(X_{t_k} = i_k \mid X_0 = i_0, \ldots X_{t_{k-1}} = i_{k-1}) = p_{i_{k-1} i_k}(t_k - t_{k-1})$$

$j \ge i$

$$p_{ij}(t) = \mathbb{P}_i(X_t = j) \equiv p_{0,j-i}(t)$$

$$\parallel$$

$$\mathbb{P}(J_0 \le J_1 \ldots \le J_{j-i} < t)$$

Proof (b) $\Rightarrow$ (c)

$$\begin{cases} p_{i0}'(t) = -q_0 p_{i0}(t), & p_{i0}(0) = \delta_{i0} \\ p_{ij}'(t) = q_{j-1} p_{i,j-1}(t) - q_j p_{ij}(t), & p_{ij}(0) = \delta_{ij} \end{cases}$$

i.e. $\quad P'(t) = P(t) Q, \quad P(0) = I$

$p_{ij}(t)$ — exists but cannot give formulae

$$Q = \begin{bmatrix} -q_0 & q_0 & & \\ & -q_1 & q_1 & \\ & & \ddots & \ddots \end{bmatrix}$$

Theorem $\{X_t\}_{t\geq 0}$ be a right continuous process with values in a finite set S.
Let Q be a Q-matrix on S & Π be its jump-matrix on S.
The following are equivalent. ["definitions"]

(a) [Jump chain / Holding time] - Conditional on $X_0 = i$, the jump chain $(Y_n)_{n\geq 0}$ of $(X_t)_{t\geq 0}$ is a discrete time Markov chain on S with transition matrix Π & initial distribution δ_i. For $n \geq 1$ conditioned on $Y_0, \ldots, Y_{n-1}$, the holding times $S_1 \ldots S_n$ are independent exponential random variables with parameters $q(Y_0), q(Y_1), \ldots, q(Y_{n-1})$ respectively.

(b) [Infinitesimal definition] for all $t, h \geq 0$

conditional on $X_t = i$, $(X_{t+\ell})_{\ell \geq 0}$ is independent of $(X_s : s \leq t)$

& as $h \downarrow 0$ uniformly in t for all j

$$\mathbb{P}(X_{t+h} = j \mid X_t = i) = \delta_{ij} + q_{ij} h + o(h)$$

(c) [Transition Probability] for all $n = 0, 1, 2, \ldots$

& $0 \leq t_0 \leq t_1 \ldots \leq t_{n+1}$ & $i_0, \ldots i_{n+1} \in S$

$$\mathbb{P}(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \ldots, X_{t_n} = i_n) = p_{i_n, i_{n+1}}(t_{n+1} - t_n)$$

$$\boxed{p_{ij}(t) = \mathbb{P}(X_t = j \mid X_0 = i)}$$

D where

Definition

Contin

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where $\{p_{ij}(t) : i,j \in S, t \geq 0\}$ is a solution to the forward equation

$$P'(t) = P(t)Q \quad , \quad P(0) = I$$

Definition: $(X_t)_{t\geq 0}$ minimal right continuous process on S ($|S| < \infty$) is called a continuous time Markov chain on S with generator matrix Q if it satisfies (a) (b) or (c) in Theorem.

Theorem Let $\{X_t\}_{t\geq 0}$ be a right continuous process with values on countable set S. Let Q be a Q-matrix on S & Π be its jump-matrix on S. The following are equivalent ["definitions"]

(a) [Jump chain / Holding time] - Conditional on $X_0 = i$, the jump chain $(Y_n)_{n\geq 0}$ of $(X_t)_{t\geq 0}$ is a discrete time Markov chain on S with transition matrix Π & initial distribution δ_i. For $n \geq 1$ conditioned on $Y_0, \ldots Y_{n-1}$, the holding times $S_1 \ldots S_n$ are independent exponential random variables with parameter $q(Y_0), q(Y_1), \ldots, q(Y_{n-1})$ respectively.

No change from $|S| < \infty$

(b) $n = 0, 1, 2, \ldots, \; 0 \le t_0 < t_1 \ldots \le t_{n+1}$ & $i_0, \ldots i_{n-1} \in S$

$$P(X_{t_{n+1}} = i_{n+1} \mid X_0 = i_0, \ldots, X_{t_n} = i_n) = p_{i_n, i_{n+1}}(t_{n+1} - t_n)$$

where $P(t) = [p_{ij}(t)]$ is a solution to backward equation

$$P'(t) = Q\,P(t), \quad P(0) = I$$

$n-1 \in S$

$n)$

$P(0) = I$

Model 1 :- (Γ, μ) - weighted graph

$\Gamma = (V, E)$, locally finite & connected

$\mu_{xy} = \mu_{yx} \neq 0 \iff \{y, x\} \in E$

$0 \neq \mu_x = \sum_{y \sim x} \mu_{xy} \quad \forall x \in V$

$\{X_n\}_{n \geq 0}$ - [random walk on (Γ, μ)] - Markov chain on V

with transition matrix $[P_{xy}]_{x,y \in V}$ $\quad P_{xy} = \dfrac{\mu_{xy}}{\mu_x}$

check :- $\mu_x P_{xy} = \mu_y P_{yx}$ - Suppose $\mu(V) = \sum_{x \in V} \mu_x < \infty$ then

$\nu_x := \frac{\mu_x}{\mu(V)}$ defines a Probability on V & $\{\nu_i\}$ is a reversible measure for $(Y_n)_{n\geq 0}$

let $\{N_t\}_{t\geq 0}$ — poisson process with rate λ.

$X_t := Y_{N_t}$ is a continuous time markov chain on V

Ex:- what is Q?

what is $p_{ij}(t)$?