Recall:

Theorem 1.9.14 $\{X_n\}_{n\geq 0}$ is a Markov chain on S with irreducible transition matrix P.

$\{X_n\}_{n\geq 0}$ is positive recurrent $\Longleftrightarrow$ $\{X_n\}_{n\geq 0}$ has a stationary distribution

$$\pi(i) = \frac{1}{E_i(T_i)}$$

Proof. $\boxed{\Leftarrow}$ ✓

$\boxed{\Rightarrow}$ shown. $\pi(i) = \dfrac{1}{E_i[T_i]}$

(i) $\sum_{i \in S} \pi$

(ii) $\sum_{i \in S}$

set $\sum_{i \in S} \pi($

So $\hat{\pi}(\cdot)$

($\Leftarrow$ Proof)

[already knew if

$\Rightarrow$) $\pi(i) = \dfrac{1}{E_i(T_i)}$

stationary

(T_i)

(i) $\sum_{i \in S} \pi(i) \le 1$

(ii) $\sum_{i \in S} \pi(i) p_{ij} = \pi(j)$

set $\sum_{i \in S} \pi(i) = c > 0$. let $\tilde{\pi}(i) = \frac{\pi(i)}{c}$

So $\tilde{\pi}(\cdot)$ is a stationary distribution for $(X_n)_{n \ge 0}$

($\Leftarrow$ Proof) $\Rightarrow$ $c = 1$

[already knew if $\pi(\cdot)$ were stationary distribut.,
$\Rightarrow \pi(j) = \frac{1}{E_j(T_j)}$ $\quad \frac{N_n(j)}{n} \to \frac{1}{E_j(T_j)} \overset{(DCT)}{\Rightarrow} \frac{\sum_{k=1}^{n} p_{ij}^k}{n} \to \frac{1}{E_j(T_j)}$]

$\square$

Thm 1.9.11:- let $(X_n)_{n \ge 0}$ be a Markov chain on S with transition matrix P. Assume that $(X_n)_{n \ge 0}$ is aperiodic & irreducible. If $(X_n)_{n \ge 0}$ has a stationary distribution, π, then

$$\lim_{n \to \infty} p_{ij}^n = \pi(j) \quad \forall i \in S, j \in S$$

Proof:-

Define:-

transi

$T^b = \inf$

step 1 (Z

Proof :- Let μ be the initial distribution of X_n.
Let $(Y_n)_{n\geq 0}$ be another Markov chain on S with transition matrix P and initial distribution π & independent $(X_n)_{n\geq 0}$.

Define :- $Z_n = (X_n, Y_n)$. $(Z_n)_{n\geq 0}$ is a Markov chain on $S\times S$ with transition matrix $\tilde{P}$ such that $\tilde{p}_{\tilde{i},\tilde{j}} = p_{i_1 j_1} p_{i_2 j_2}$ $\quad \tilde{i} = (i_1, i_2)$, $\tilde{j} = (j_1, j_2)$
(Exercise).

$$T^b = \inf\{n\geq 1 \mid Z_n = (b,b)\} \text{ for } b\in S.$$

Step 1. $(Z_n)_{n\geq 0}$ is positive recurrent claim:- P – aperiodic & irreducible $\tilde{i} \;\&\; \tilde{j} \in S \times S$

$\exists\, n_0 \geq 1$ such $\begin{cases} p^n_{i_1 j_1} > 0 & \forall n \geq n_0 \\ p^n_{i_2 j_2} > 0 & \forall n \geq n_0 \end{cases}$

Proof:- let $k \in S$ such that $d(k) = 1$.

(irreducibility) $\exists\, r, s > 0$ such that $p^r_{i,k} > 0$, $p^s_{k j_1} > 0$

(aperiodicity) $\exists\, n_1 > 0$ such $p^n_{kk} > 0 \;\forall n \geq n_1$

(Chapman-Kolmogorov) $\Rightarrow$ $p^{r+n+s}_{i_1 j_1} > 0 \;\forall n \geq n_1$ □

Claim $\Rightarrow \exists\, n_0 \geq 1$

$\tilde{P}^n_{\tilde{i}\,\tilde{j}} = p^n_{i_1 j_1}\, p^n_{i_2 j_2} > 0 \quad \forall \tilde{i}, \tilde{j} \in S \times S$

$\forall n \geq n_0$

$\therefore \tilde{P}$ irreducible & aperiodic – (*)

furthermore

$i \in S \times$

$\Rightarrow$

Thm 1.

further $\tilde{\pi}(\tilde{i}) = \pi(i_1)\pi(i_2) \quad \forall \tilde{i} \in S \times S$

$$\sum_{\tilde{i} \in S\times S} \tilde{\pi}(\tilde{i})\, \tilde{p}_{\tilde{i}\tilde{j}} = \left(\sum_{i_1 \in S} \pi(i_1)\, p_{i_1 j_1}\right)\left(\sum_{i_2 \in S} p_{i_2 j_2}\, \pi(i_2)\right)$$

$$= \pi(j_1)\, \pi(j_2)$$

π stationary for $(X_n)_{n\geq 0}$

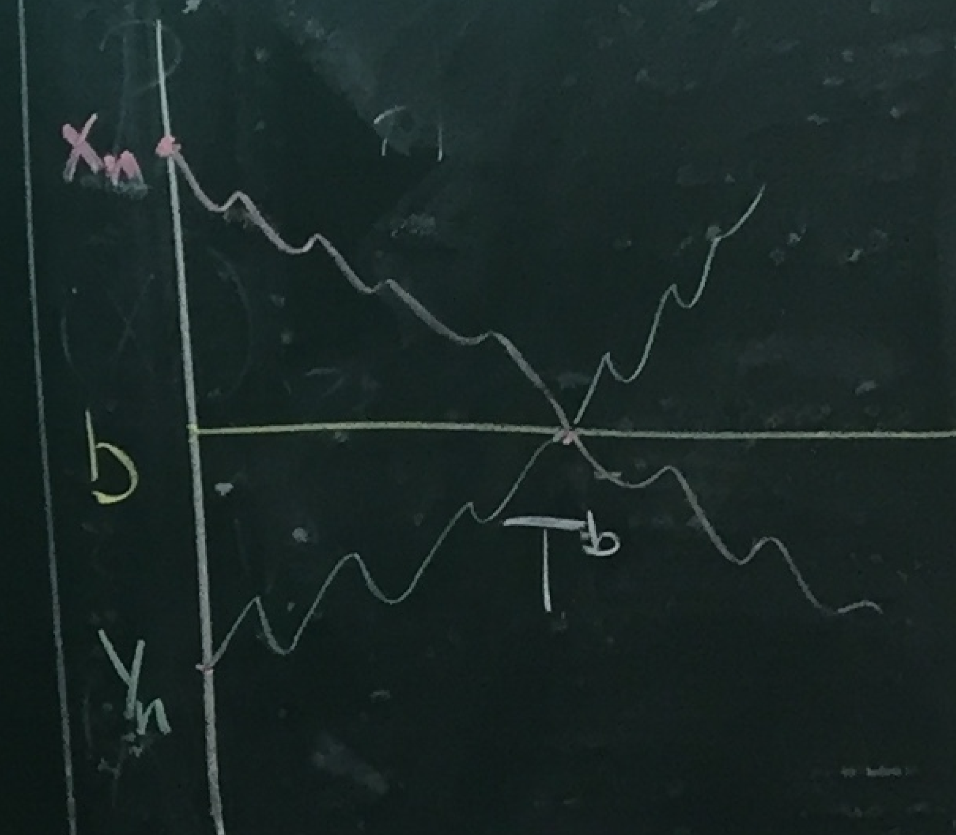
$$= \tilde{\pi}(\tilde{j}) \quad \forall \tilde{j} \in S \times S.$$

$\Rightarrow \tilde{\pi}(\cdot)$ is stationary for $(Z_n)_{n \geq 0}$

Thm 1.9.14 : $(Z_n)_{n\geq 0}$ is positive recurrent

$\Rightarrow E[T^b] < \infty$

$\Rightarrow \mathbb{P}(T^b < \infty) = 1$



$$W_n = \begin{cases} X_n & n \le T^b \\ Y_n & T^b \le n \end{cases}$$

$(Z_n)_{n\ge 0}$ is M.C on $S\times S$, $\mathbb{P}(Z_0 = \underset{\sim}{i}) = \mu(i_1)\,\pi(i_2)$

$\mathbb{P}(T^b < \infty) = 1$

Thm 1.8.2 (Strong Markov Property)

- $Z_{T^b} = (b,b)$

$(Z_{T^b+n})_{n\ge 0}$ is a MC on $S\times S$ with transition matrix $\tilde{P}$ and initial distribution $\delta_{(b,b)}$ & independent of $(Z_k)_{k\le T^b}$

$(\ddagger)$ $(X_{T^b+n}, Y_{T^b+n})_{n \geq 0}$ is a Markov chain on $S \times S$

. initial distribution $\delta_{(b,b)}$

. Transition matrix $\tilde{P}$

& independent of

$(X_0, Y_0), \ldots, (X_{T^b}, Y_{T^b})$

By Symmetry

$(\ddagger\ddagger)$ $(Y_{T^b+n}, X_{T^b+n})_{n \geq 0}$ is a Markov chain on $S \times S$

. initial distribution $\delta_{(b,b)}$

. transition matrix $\tilde{P}$

. & independent of

$(X_0, Y_0), \ldots, (X_{T^b}, Y_{T^b})$

$P(T^b < \infty)$

$(\ddagger)$

$(\ddagger\ddagger)$

$$W'_n = \begin{cases} Y_n & n \le T^b \\ X_n & T^b \le n \end{cases}$$

$$\text{Set } V_n = (W_n, W'_n) \equiv \begin{cases} (X_n, Y_n) & n \le T^b \\ (Y_n, X_n) & T^b \le n \end{cases}$$

$$V_1, \ldots, V_k, \ldots, V_{T^b}, V_{T^b+1}, \ldots V_{T^b+n}$$

$$|||$$

$$(X_1, Y_1), \ldots (X_k, Y_k)\ (X_{T^b}, Y_{T^b}),\ (Y_{T^b+1}, X_{T^b+1}) \ldots (Y_{T^b+n}, X_{T^b+n})$$

$$\|_d$$

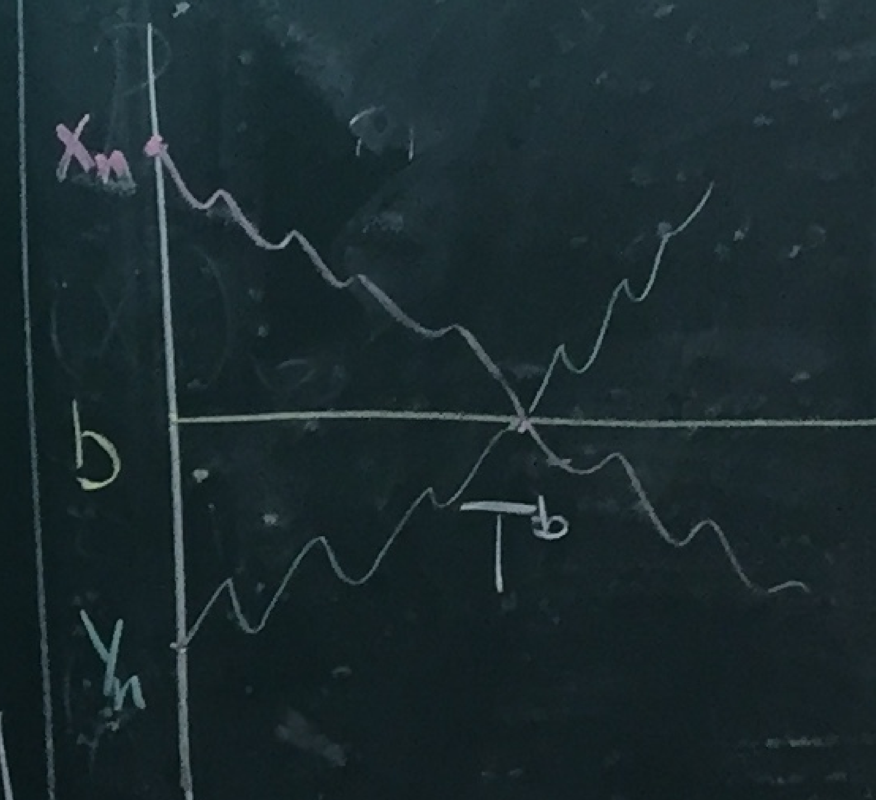
$$(X_1, Y_1)\ (X_k, Y_k)\ (X_{T^b}, Y_{T^b}) \ldots (X_{T^b+1}, Y_{T^b+1}) \ldots (X_{T^b+n}, Y_{T^b+n})$$

$P(T^b < \infty) = 1$

$\boxed{\text{Ex}}$

$\Rightarrow E[T^b] < \infty$

$\Rightarrow \mathbb{P}(T^b < \infty) = 1$



$$W_n = \begin{cases} X_n & n \le T^b \\ Y_n & T^b \le n \end{cases}$$

$\Rightarrow$ V_n is a M.C on $S \times S$ with transition matrix $\tilde{P}$, on $S \times S$.

$\Rightarrow$ $(W_n)_{n \geq 0}$ is a M.C on S with transition matrix P & $\mathbb{P}(W_0 = i) = \mathbb{P}(X_0 = i)$
$= \mu(\{i\})$

$$|\mathbb{P}(X_n = j) - \pi(j)| = |\mathbb{P}(X_n = j) - \mathbb{P}(Y_n = j)| = |\mathbb{P}(W_n = j) - \mathbb{P}(Y_n = j)|$$

$$= |\mathbb{P}(X_n = j \cap n \leq T^b) + \mathbb{P}(Y_n = j \cap T^b \leq n) - \mathbb{P}(Y_n = j)|$$

$$= |\mathbb{P}(X_n = j \cap n \leq T^b) - \mathbb{P}(Y_n = j \cap n \leq T^b)|$$

$$\leq \mathbb{P}(X_n = j \cap n \leq T^b) + \mathbb{P}(Y_n = j \cap n \leq T^b) \leq \mathbb{P}(n \leq T^b)$$

Markov Inequality

$$\mathbb{P}(n \le T^b) \le \frac{E[T^b]}{n} \longrightarrow 0 \quad \text{as } n \to \infty$$

Take $\mu = \delta_i$ for $i \in S$

$$\Rightarrow \quad |p^n_{ij} - \pi(j)| \longrightarrow 0 \quad \forall j \in S. \quad \square$$

We have shown: $(X_n)_{n \ge 0}$ M.C on S, irreducible & aperiodic & has stationary distribution π $|\mathbb{P}(X_n = j) - \pi(j)| \to 0$ as $n \to \infty$

$W_n^1 = \{$

Set $V_n =$

$V_1, \ldots, V_k,$

$(X_1, Y_1), \ldots (X_k$

$P(T^b < \infty) = 1$ & (‡)

(‡)

$(X_1, Y_1)\ (X_k$

Remark: (Another Proof of uniqueness for stationary distribution π)

$(X_n)_{n\geq 0}$ is a Markov chain on S and transition matrix P. Suppose P is irreducible and has two stationary distributions π and η

let $\varphi = \dfrac{I+P}{2}$, φ is irreducible & Aperiodic (Ex.)

Fubini $\pi \varphi = \varphi$ $\Rightarrow$ π, η are stationary for φ

$\eta \varphi = \varphi$

Thm 1.9.1) $\Rightarrow$ Fix $i \in S$, $q^n_{ij} \longrightarrow \pi(j)$ as $n \to \infty$

$q^n_{ij} \longrightarrow \eta(j)$ as $n \to \infty$ $\forall j \in S$

$\Rightarrow \pi = \eta.$

Ergodic Thm

$(X_n)_{n\geq 0}$ is a Markov chain on S with initial distribution μ and irreducible transition matrix P. Suppose $(X_n)_{n\geq 0}$ is positive recurrent. $f: S \to \mathbb{R}$ be a bounded function

$$\frac{1}{n}\sum_{k=1}^{n} f(X_k) \longrightarrow \sum_{s\in S} \pi(s) f(s) \quad \text{w.p. } 1$$

where $\pi(\cdot)$ is the unique stationary distribution given by

$$\pi(i) = \frac{1}{E_i[T_i]} \quad \forall i \in S$$