Recall

$(X_n)_{n \geq 0}$ is a Markov chain on S with transition matrix P

$$V_i(n) = \sum_{k=0}^{n-1} 1(X_k = i)$$

Theorem [illegible]
$$\begin{cases} \text{Assume } (X_n)_{n \geq 0} \text{ is irreducible} \\ \lim_{n \to \infty} \frac{V_i(n)}{n} = \frac{1}{E_i(T_i)} \quad \text{w.p. } 1 \end{cases}$$

Remark
$$\begin{cases} i \text{ is transient} & P_i(T_i = \infty) > 0 \\ i \text{ is null recurrent} & P_i(T_i < \infty) = 1,\ E_i(T_i) = \infty \\ i \text{ is positive recurrent} & P_i(T_i < \infty) = 1,\ E_i(T_i) < \infty \end{cases}$$

Goal: $\begin{cases} \cdot \text{ stationary distribution } \pi < \begin{matrix}\text{uniqueness} \\ \text{Existence}\end{matrix} \\ \cdot \text{ i.e. } \; p_{ij}^{n} \longrightarrow \pi(j)\,(?) \quad \forall j \in S \end{cases}$

(Dominated Convergence Theorem): $\{X_n\}_{n\geq 0}$ is a sequence of random variables

Such that (i) $|X_n| \leq Z$ & $E[Z] < \infty$

(ii) $X_n \longrightarrow X$ with probability one $= \begin{cases} A = \{\omega \in \Omega \mid \lim_{n\to\infty} X_n(\omega) = X(\omega)\} \\ \mathbb{P}(A) = 1 \end{cases}$

$\boxed{\text{Assume this as fact}}$

Then $E[X_n] \longrightarrow E[X]$ as $n \to \infty$

Observe $0 \leq N_n(i) \leq n \implies 0 \leq \frac{N_n(i)}{n} \leq 1$ — (*)

$j \in S$

$$E_j\left[\frac{N_n(i)}{n}\right] = E_j\left[\frac{\sum_{k=1}^{n} \mathbb{1}(X_k = i)}{n}\right] = \frac{\sum_{k=1}^{n} E_j[\mathbb{1}(X_k = i)]}{n} = \frac{\sum_{k=1}^{n} p_{ji}^{k}}{n}$$

Apply Dominated convergence Theorem to: $X_n = \frac{N_n(i)}{n}$, $X = \frac{1}{E_i[T_i]}$, $Z = 1$

Fix $i \in S$

$$\Rightarrow \forall j \in S \quad \frac{\sum_{k=1}^{n} p_{ji}^{k}}{n} \longrightarrow \frac{1}{E_i[T_i]}$$

$$\Downarrow$$

$$\boxed{E_j[X] = E_j\left[\frac{1}{E_i[T_i]}\right] = \frac{1}{E_i[T_i]}}$$

Theorem | 9.14 $(X_n)_{n \geq 0}$ is an irreducible Markov chain on S with transition matrix.

$(X_n)_{n \geq 0}$ has a stationary distribution $\Leftrightarrow$ Every state is positive recurrent

π & $\pi(i) = \frac{1}{E_i(T_i)}$

Proof: $\boxed{\Rightarrow}$ let $(X_n)_{n\geq 0}$ have a stationary distribution π

Thm 1.9.8: & DCT

Fix $i \in S$, $\frac{1}{n}\sum_{k=1}^{n} p_{ji}^{k} \longrightarrow \frac{1}{E_i[T_i]}$ $\forall j \in S$

$\exists$ $j_0 \in S$ such that $\pi(j_0) > 0$

π-stationary $\Rightarrow$ $\pi(j_0) = \sum_{j\in S} \pi(j)\, p_{jj_0}^{k}$ $\forall k \geq 1$

$\Rightarrow \forall n \geq 1$ $\pi(j_0) = \sum_{j\in S} \pi(j) \frac{\sum_{k=1}^{n} p_{jj_0}^{k}}{n}$

$Y_n(j) = \sum$

D.C.T

let $j \in S$

$$Y_n(j) = \frac{\sum_{k=1}^{n} p^k_{j i_0}}{n}, \quad Y = \frac{1}{E_{i_0}[T_{i_0}]}, \quad 0 \le Y_n(j) \le 1 \equiv Z$$

$$E_\pi[Y_n(\cdot)] = \sum_{j \in S} \pi(j)\, Y_n(j) = \sum_{j \in S} \pi_j \frac{\sum_{k=1}^{n} p^k_{j i_0}}{n} \equiv \pi(i_0)$$

D.C.T

$$\downarrow$$

$$\frac{1}{E_{i_0}[T_{i_0}]} = E_\pi[Y]$$

We have shown $\pi(i_0) = \dfrac{1}{E_{i_0}[T_{i_0}]}$. As $\pi(i_0) > 0$, $E_{i_0}[T_{i_0}] < \infty$ $\Rightarrow$ $i_0 \in S$ is positive recurrent.

let $j \in S$. $i_0 \longleftrightarrow j$ as $(X_n)_{n \ge 0}$ is irreducible.

$\Rightarrow \exists\, n_0$ such that $p^{n_0}_{i_0 j} > 0 \Rightarrow \Pi(j) = \sum_{k \in S} \Pi(k)\, p^{n_0}_{kj} \geq \Pi(i_0)\, p^{n_0}_{i_0 j} > 0$

$\Rightarrow \Pi(j) > 0$

By the argument used for $i_0 \in S$, we conclude $j \in S$ is also positive recurrent & $\Pi(j) = \dfrac{1}{E_j[T_j]}$.

$\Rightarrow$ all states $j \in S$ are positive recurrent □

(This is a proof for $\boxed{\text{uniqueness}}$ of stationary distribution Π for irreducible Markov chain → "At most one")

$\boxed{\Leftarrow}$ Given all states are positive recurrent

$j \in S \quad E_j[T_j] < \infty$

let $\Pi(j) = \dfrac{1}{E_j[T_j]}$ for all $j \in S$ $\Rightarrow 0 < \Pi(j) \leq 1$

Fix $i \in S$ $\quad \sum_{j\in S} p_{ij}^{n} = 1 \quad \forall n \geq 1$

$$\Rightarrow \frac{\sum_{k=1}^{n} \sum_{j\in S} p_{ij}^{k}}{n} = 1 \quad \forall n \geq 1$$

let $S' \subseteq S$ $\quad |S'| < \infty$. The above implies

$$\sum_{j\in S'} \frac{\sum_{k=1}^{n} p_{ij}^{k}}{n} = \frac{\sum_{k=1}^{n} \sum_{j\in S'} p_{ij}^{k}}{n} \leq \frac{\sum_{k=1}^{n} \sum_{j\in S} p_{ij}^{k}}{n} = 1$$

$(*)$

As $\lim_{n\to\infty}$

$(*) \Rightarrow$

$\Rightarrow$

$S_n \subseteq S \quad |S_n| < \infty$

$\bigcup_{n=1}^{\infty} S_n = S \quad S_n \subseteq S_{n+1}$

$\sum_{j\in S_n} \pi(j) \leq 1 \quad \Rightarrow$

$\downarrow n\to\infty$

$\sum_{j\in S} \pi(j) \leq 1$

$$\text{As} \quad \lim_{n\to\infty} \frac{\sum_{k=1}^{n} p_{ij}^{k}}{n} = \frac{1}{E_j[\tau_j]} \quad \& \quad |S'| < \infty$$

$$(*) \implies \sum_{j\in S'} \frac{1}{E_j[\tau_j]} \le 1 \quad \forall\, S' \subseteq S,\ |S'| < \infty$$

$$\implies \sum_{j\in S'} \pi(j) \le 1 \quad \forall\, S' \subseteq S \quad |S'| < \infty$$

$S_n \subseteq S \quad |S_n| < \infty$

$S_n \subseteq S_{n+1}$

$\bigcup_{n=1}^{\infty} S_n = S$

$\sum_{j\in S_n} \pi(j) \le 1$

$\downarrow$ as $n\to\infty$

$\sum_{j\in S} \pi(j) \le 1$

$$\implies \boxed{\sum_{j\in S} \pi(j) \le 1} \quad (1)$$

$$\frac{\sum_{k=1}^{n} \sum_{j\in S} p_{ij}^{k}}{n} = 1$$

$(*)$

Fix $i \in S$,

$\implies$

$\implies$

let $S' \subseteq$

Fix $i \in S, j \in S$

$$p_{ij}^{k+1} = \sum_{l \in S} p_{il}^{k} p_{lj}$$

$$\Rightarrow \sum_{l \in S} \left(\frac{\sum_{k=1}^{n} p_{il}^{k}}{n} \right) p_{lj} = \frac{1}{n} \sum_{k=1}^{n} p_{ij}^{k+1}$$

$$\Rightarrow \sum_{l \in S} \left(\frac{\sum_{k=1}^{n} p_{il}^{k}}{n} \right) p_{lj} = \frac{1}{n} \sum_{k=1}^{n+1} p_{ij}^{k} - \frac{1}{n} p_{ij}$$

let $S' \subseteq S$, $|S'| < \infty$

$$\sum_{l \in S'} \left(\frac{\sum_{k=1}^{n} p_{il}^{k}}{n} \right) p_{lj} \leq \frac{1}{n} \sum_{k=1}^{n+1} - \frac{1}{n} p_{ij}$$

Let $n \to \infty$, we have

$$\sum_{\ell \in S'} \pi(\ell) p_{\ell j} \leq \pi(j) \quad \forall S' \subseteq S, \ |S'| < \infty$$

Similarly $\Longrightarrow$

$$\sum_{\ell \in S} \pi(\ell) p_{\ell j} \leq \pi(j) \quad \forall j \in S \quad (**)$$

Suppose

$$\sum_{\ell \in S} \pi(\ell) p_{\ell j_0} < \pi(j_0) \quad \text{for some } j_0 \in S \quad (***)$$

(**) & (***) $\Longrightarrow$

$$\sum_{j \in S} \sum_{\ell \in S} \pi(\ell) p_{\ell j} < \sum_{j \in S} \pi(j)$$

$$\sum_{j \in S} p_{\ell j} = 1 \Longrightarrow$$

$$\Rightarrow \sum_{l\in S} \pi(l) \sum_{j\in S} p_{lj} < \sum_{j\in S} \pi(j)$$

$\sum_{j\in S} p_{lj} = 1$

$$\Rightarrow \sum_{l\in S} \pi(l) < \sum_{j\in S} \pi(j)$$

Contradiction

$$\boxed{\sum_{l\in S} \pi(l) p_{lj} = \pi(j) \quad \forall j \in S.} \quad (2)$$

$$(1) \;\&\; (2) \Rightarrow \sum_{j\in S} \pi(j) = 1$$

Fix $i \in S$,

$\Rightarrow$

$\Rightarrow$

let $s' \in S$,