General theory for cnt. time MCs

S countable set. Recall Q matrix on S

- $0 \leq -q_{ii} < \infty$ for all i $\qquad q_i = -q_{ii}$
- $q_{ij} \geq 0$ for all $i \neq j$
- $\sum_{j \in S} q_{ij} = 0$ for all i

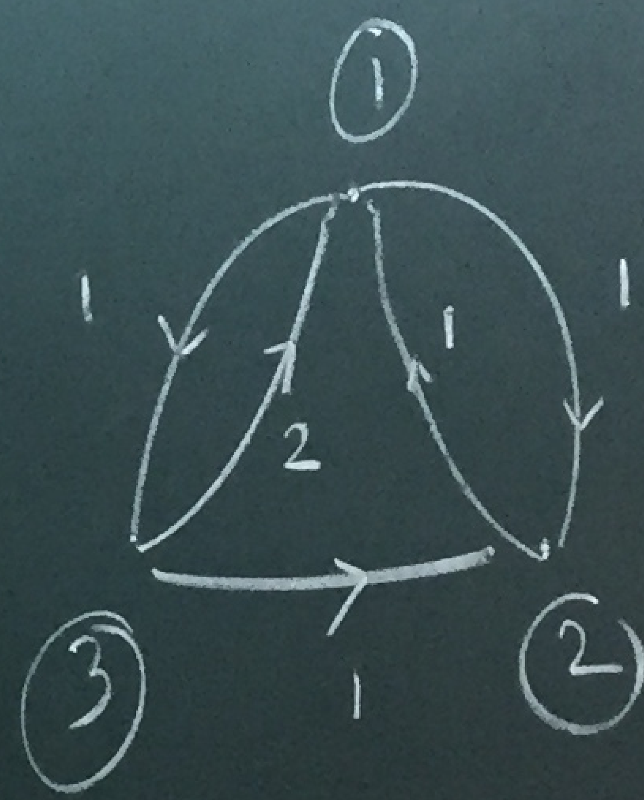
Def. The <u>jump matrix</u> $\Pi = (\pi_{ij},\ i,j \in S)$ of Q is

$$\pi_{ij} = \begin{cases} \dfrac{q_{ij}}{q_i} & \text{if } j \neq i \ \& \ q_i \neq 0 \\ 0 & \text{if } j \neq i \ \& \ q_i = 0 \end{cases}$$

$$\pi_{ii} = \begin{cases} 0 & \text{if } q_i \neq 0 \\ 1 & \text{if } q_i = 0 \end{cases}$$

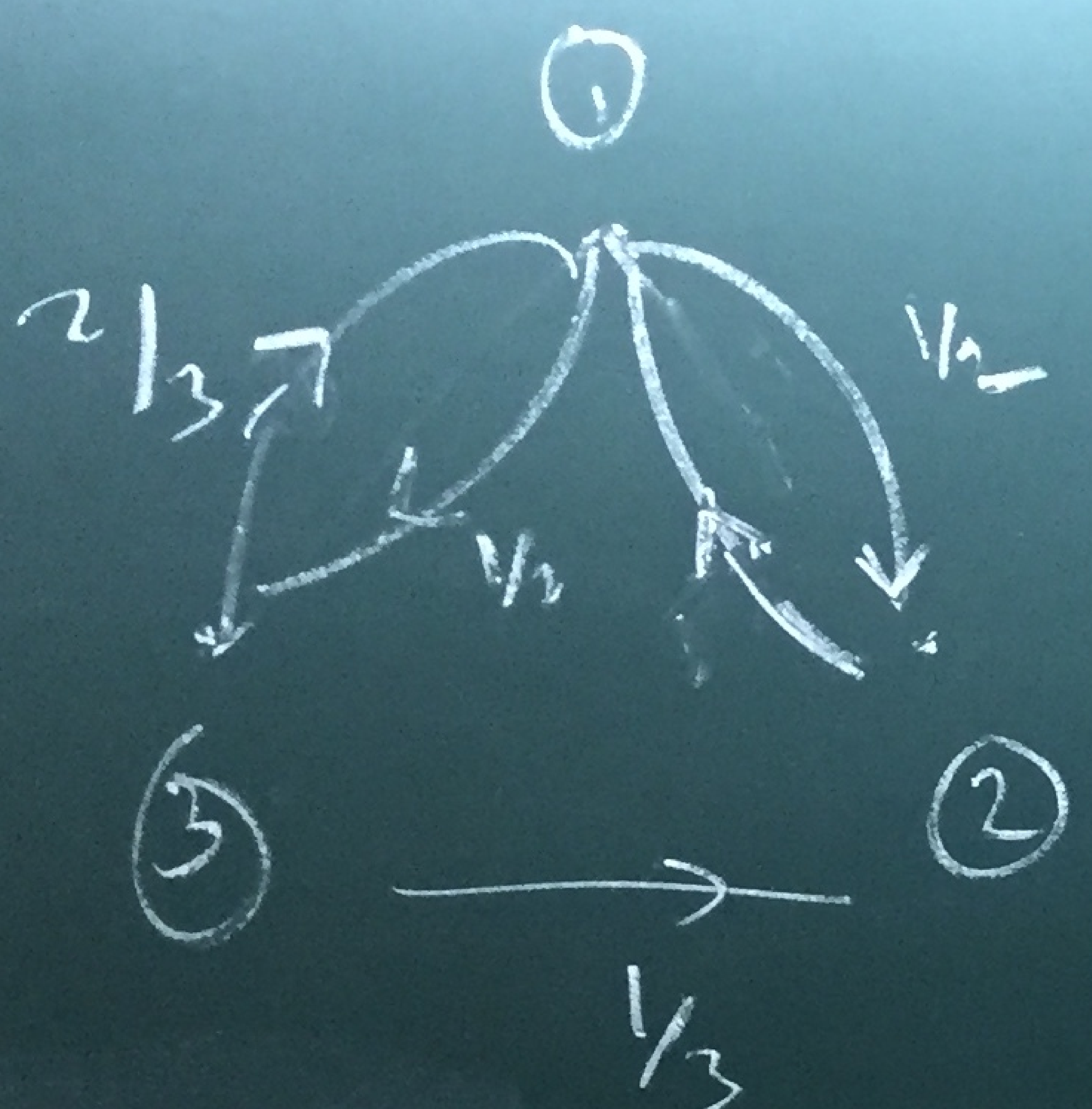
Eg.

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}$$

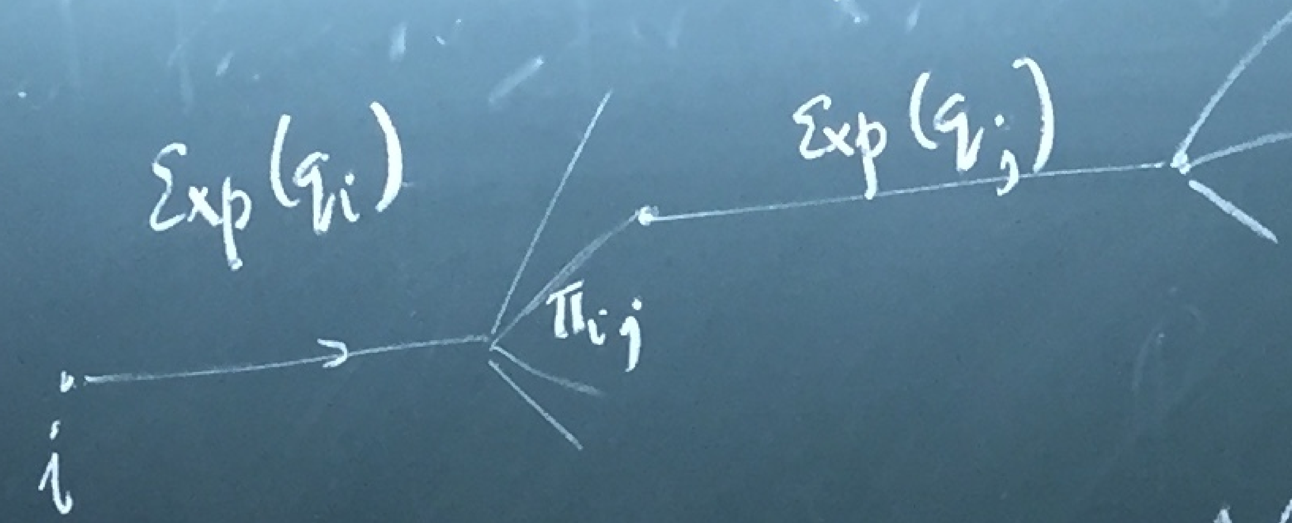


Jump matrix

$$\Pi = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$



Def. A minimal right cont process $(X_t)_{t \geq 0}$ on S is a Markov chain with initial distribution λ & generator matrix Q if its jump chain $(Y_n)_{n \geq 0}$ is a discrete time Markov(λ, Π) and if for each $n \geq 1$, conditional on $Y_0, Y_1 \dots Y_{n-1}$, its holding times $S_1, S_2 \dots S_n$ are independent Exponential random variables of parameters $q(Y_0), q(Y_1), \dots, q(Y_{n-1})$ respectively



<u>Notation</u> $(X_t)_{t \geq 0}$ is Markov(λ, Q).

<u>Construction</u> 1. Let $(Y_n)_{n \geq 0}$ be discrete Markov(λ, Π) and let $T_1, T_2, \dots \overset{\text{i.i.d.}}{\sim} \mathrm{Exp}(1)$, independent of $(Y_n)_{n \geq 0}$

Set $S_n = \dfrac{T_n}{q(Y_{n-1})} \sim \mathrm{Exp}(q(Y_{n-1}))$

(λ, Π)

dependent

$$J_n = S_1 + S_2 + \cdots + S_n$$

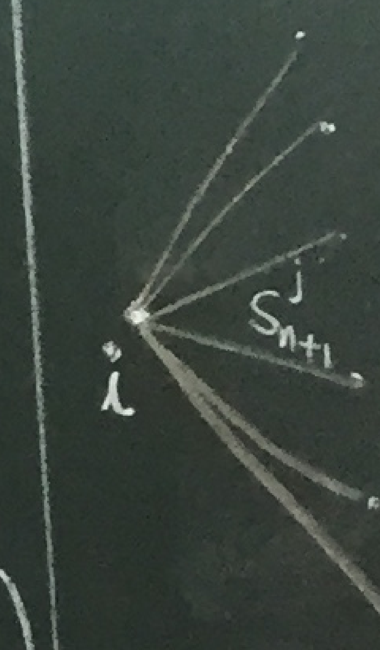
$$X_t = \begin{cases} Y_n & \text{if } J_n \le t < J_{n+1} \text{ for some } n \\ \infty & \text{otherwise} \end{cases}$$

<u>Construction 2</u>: Start with $X_0 = Y_0$ with distribution λ, and array $(T_n^j, n \geq 1, j \in S)$ of indep $\mathrm{Exp}(1)$.

Then, inductively for $n = 0, 1, \ldots$, if $Y_n = i$

let

$\mathrm{Exp}(q_{ij}) \leftarrow S_{n-}$



$\le t < J_{n+1}$ for some n

herwise.

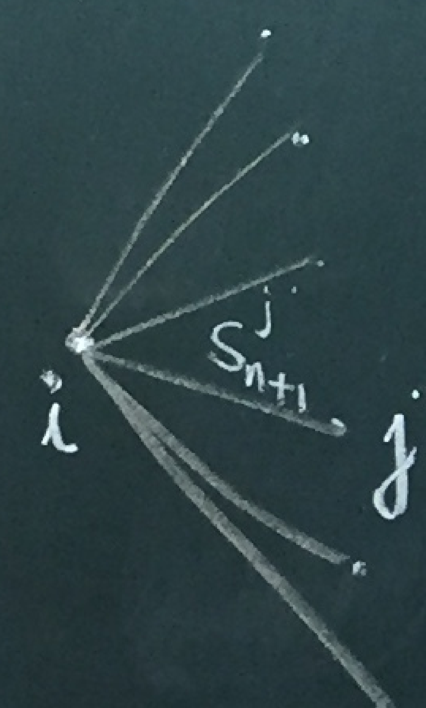
$X_0 = Y_0$ with distribution λ.

$^j_n, n \ge 1, j \in S)$ of indep $\mathrm{Exp}(1)$

$n = 0, 1, \ldots$, if $Y_n = i$

let

$$\underbrace{S^j_{n+1}}_{\mathrm{Exp}(q_{ij})} = \frac{T^j_{n+1}}{q_{ij}} \quad \text{for } j \neq i$$



$$S_{n+1} = \inf_{j \neq i} \underbrace{S^j_{n+1}}_{\mathrm{Exp}(q_i)}$$

$$Y_{n+1} = \begin{cases} j & \text{if } S^j_{n+1} = S_{n+1} < \infty \\ i & \text{if } S_{n+1} = \infty \end{cases}$$

Note Y_n

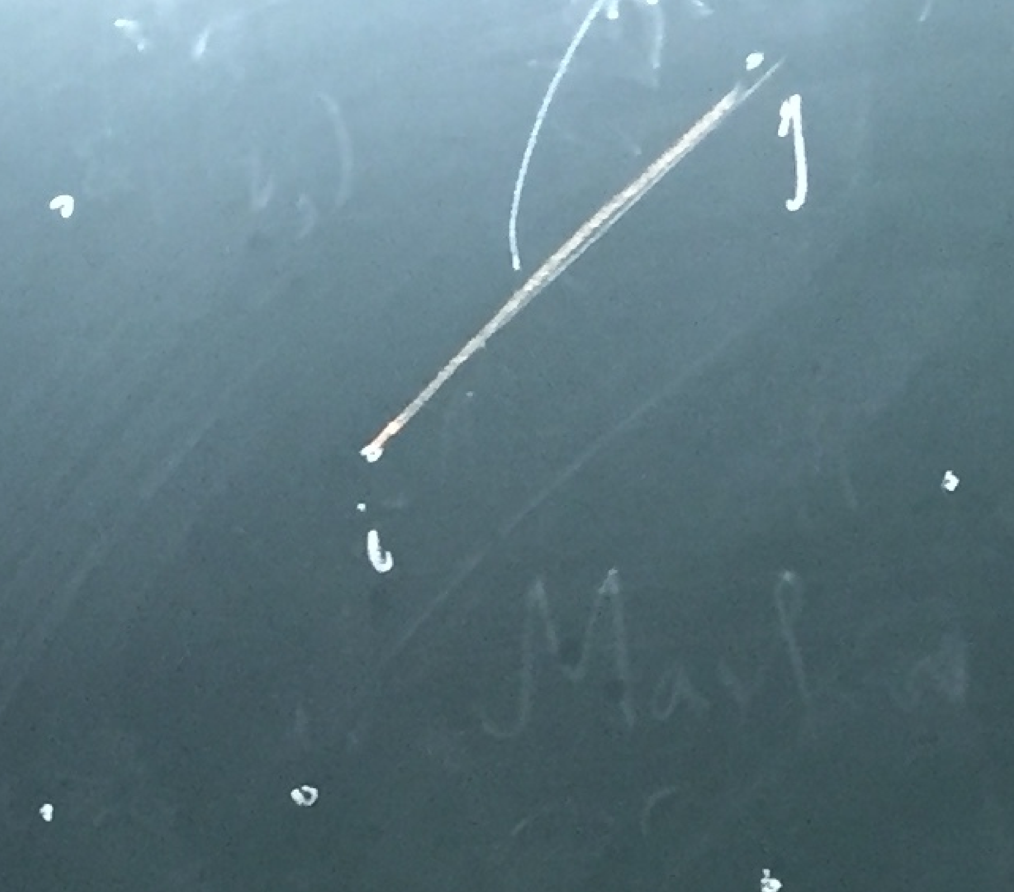
Re

Note $Y_{n+1} \sim (\pi_{ij},\ j \in S)$ and S_{n+1} & Y_{n+1} are indep (Thm 2.3.2),
and is indep of $Y_0, Y_1 \ldots Y_n$ & $S_1, \ldots S_n$

Remark: q_i : rate of leaving i

q_{ij} : rate of going from i to j

Construction 3:



$(N_t^{ij}, t \geq 0)$ Poisson process of rate q_{ij}

indep over (i, j)

Start with $X_0 = Y_0$ with distribution λ.

Set $J_0 = 0$ and inductively define for $n = 0, 1, \ldots$

$$J_{n+1} = \inf\left\{ t > J_n : N_t^{Y_n, j} \neq N_{J_n}^{Y_n, j} \text{ for some } j \neq Y_n \right\}$$

$$Y_{n+1} = \begin{cases} j & \text{if } J_{n+1} < \infty \\ i & \text{if } J_{n+1} = \infty \end{cases}$$

The first jump time of $(N_t^{ij})_{t \geq 0}$ is $\mathcal{E}xp(q_{ij})$. Conditional on

$Y_0 = i$, $J_1 \sim \mathcal{E}xp(q_i)$, Y_1 has distribution $(\pi_{ij}, j \in S)$ and J_1 & Y_1 are independent.

Suppose T is a stopping time for $(X_t)_{t \geq 0}$. If we condition on X_0 & on the processes $(N_t^{k,l})_{t \geq 0}$ for $(k,l) \neq (i,j)$, then $\{T \leq t\}$ depends only on $\{N_s^{ij}, s \leq t\}$.

So, by the strong Markov property of Poisson process $\widetilde{N}_t^{ij} = N_{T+t}^{ij} - N_T^{ij}$ ...

is a Poi...

of X...

He...

To be...

is a Poisson process of rate q_{ij} indep of $(N_s^{ij} : s \leq T)$ and indep of X_0 and $(N_t^{kl})_{t \geq 0}$ for $(k,l) \neq (i,j)$.

Hence, conditional on $T < \infty$ & $X_T = i$, $(X_{T+t})_{t \geq 0} \sim (X_t)_{t \geq 0}$ and is indep of $(X_s, s \leq T)$

To be continued by SIVA!