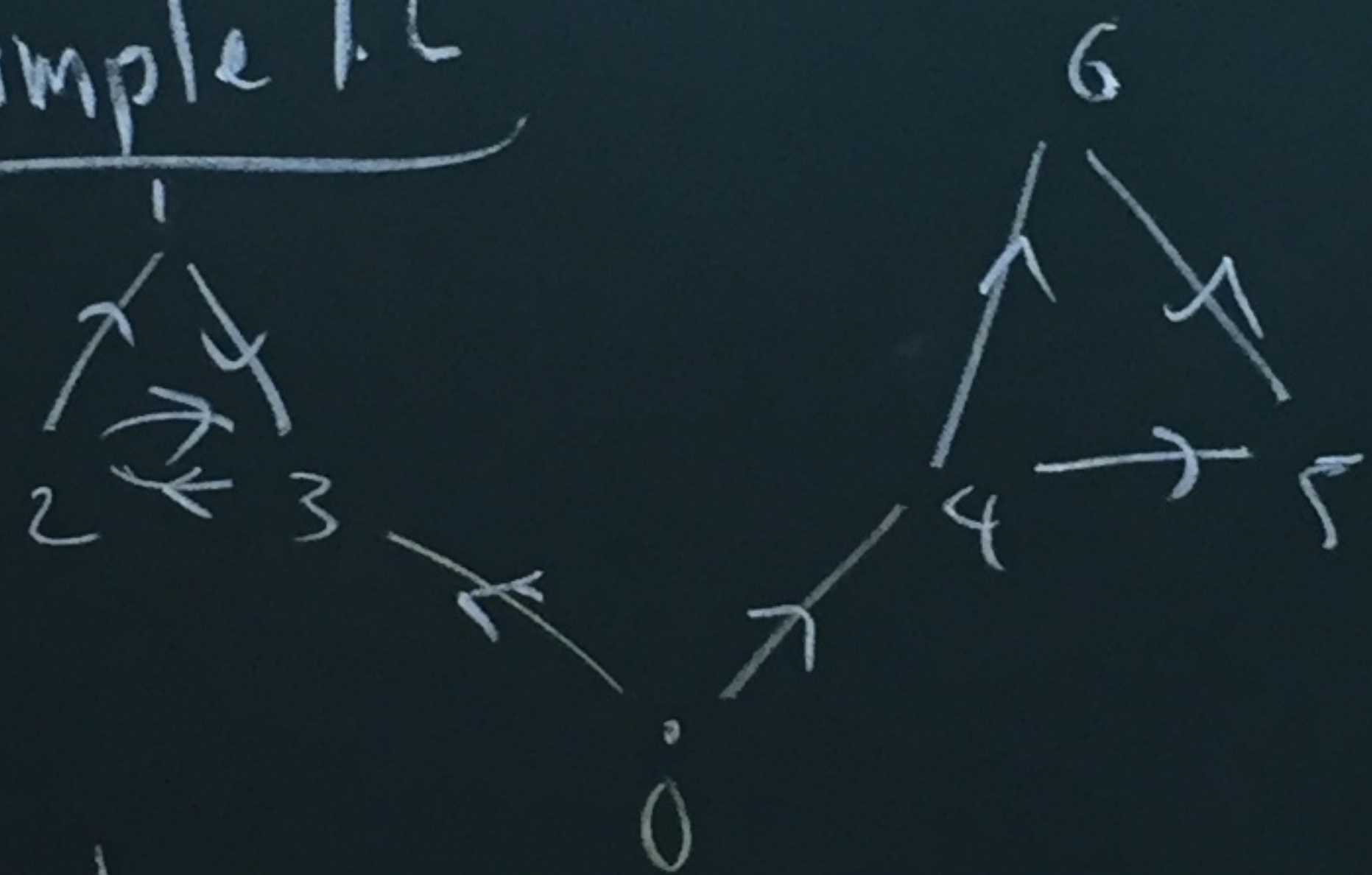


1.6 Recurrence & Transience

$(X_n)_{n \geq 0}$ is a Markov chain on S , transition matrix P and initial distribution μ .

Recall Example 1.2



of visits to 0 is either 1 or 0 in the entire life of M.C.

$i \in S$

$$N(i) = \sum_{m=1}^{\infty} 1(x_m = i)$$

$$1(x_m = i) = \begin{cases} 1 & \text{if } x_m = i \\ 0 & \text{if } x_m \neq i \end{cases}$$

max p

is the ^{total} number of visits to i by $(X_n)_{n \geq 0}$

$$N_n(i) = \sum_{m=1}^n 1(x_m = i)$$

is the number of visits to i up to time n by $(X_k)_{k \geq 0}$

For $j \in S$

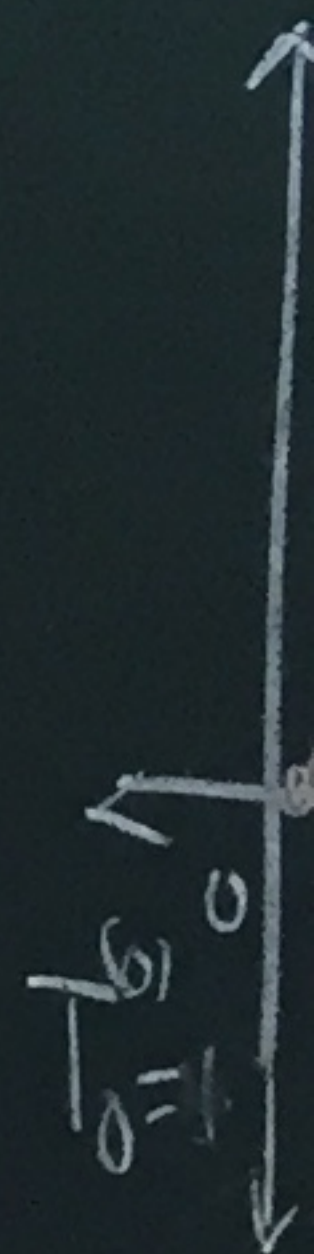
$$T^{hit}_j = \inf \{n \geq 0 \mid X_n = j\} - \text{Hitting time of } j$$

$$T_j = \inf \{n \geq 1 \mid X_n = j\} - \text{Return time to } j, \text{ under } \mathbb{P}_j(\cdot)$$

[- first passage time to state j]

Definition

Intuitive

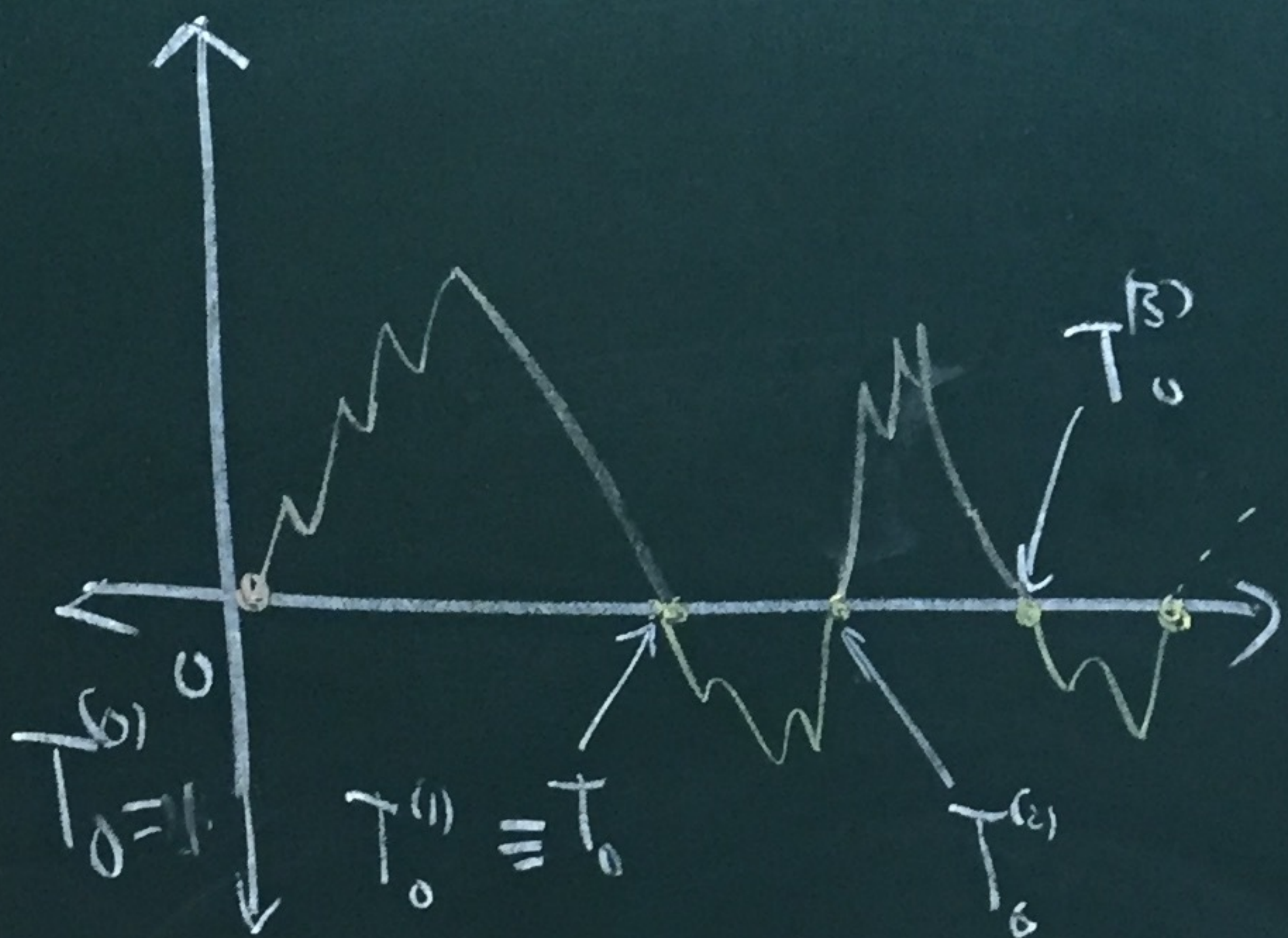


Definition 1.6.1 Let $i \in S$.

$i \in S$ is said to be recurrent if $\mathbb{P}_i(N(i) = \infty) = 1$

$i \in S$ is said to be transient if $\mathbb{P}_i(N(i) < \infty) = 1$

Intuitive observations - Simple random walk on $\mathbb{Z}$



$$T_0^{(r)} = \inf \{ i \geq T_0^{(r-1)} \mid X_i = 0 \}$$

$$S_0^r = T_0^{(r)} - T_0^{(r-1)} = i.i.d \ r \geq 1$$

Proposition 1.62: Suppose $(X_n)_{n \geq 0}$ is a Markov chain on S , with transition matrix P . A site $i \in S$ is recurrent $(\Rightarrow) \mathbb{P}_i(T_i < \infty) = 1$

Proof:- $m=1$
 $\{N(i) \geq 1\} = \{T_i < \infty\} \Rightarrow \mathbb{P}_i(N(i) \geq 1) = \mathbb{P}_i(T_i < \infty)$

$m \geq 2$ $\{N(i) \geq m\} = \{N(i) \geq m, T_i < \infty\}$
 $= \bigcup_{k=1}^{\infty} \{N(i) \geq m, T_i = k\}$

$$\begin{aligned}
 P_i(N(t) \geq m) &= \sum_{k=1}^{\infty} P_i(N(t) \geq m, T_i = k) \\
 &= \sum_{k \in N'} P_i(N(t) \geq m \mid T_i = k) P_i(T_i = k) \quad \left. \begin{array}{l} \text{if } (X_{k-1} = i) \\ \text{if } (X_{k-1} \neq i) \end{array} \right\} - (*)
 \end{aligned}$$

$$N' = \{k \geq 1 \mid P_i(T_i = k) > 0\}$$

$$= \sum_{k \in N'} P_i(N(t) \geq m \mid X_k = i, X_{k-1} \neq i, \dots, X_1 \neq i) P_i(T_i = k)$$

$$N(t) = \sum_{n=1}^{\infty} 1(X_n = i) = \sum_{n=1}^k 1(X_n = i) + \sum_{n=k+1}^{\infty} 1(X_n = i)$$

$$P_i(N(n) \geq m | T_i = k) = P_i(N(i) \geq m | X_k = i, X_{k-1} \neq i, \dots, X_1 \neq i) = P_i\left(\sum_{n=1}^k 1(X_n = i) + \sum_{n=k+1}^{\infty} 1(X_n = i) \geq m \mid X_k = i, X_{k-1} \neq i, \dots, X_1 \neq i\right)$$

$$\sum_{\substack{X_n \in A_n \\ k+1 \leq n \leq m}} P(X_{k+1} = x_{k+1}, \dots, X_m = x_m \mid X_k = i, X_{k-1} \neq i, \dots, X_1 \neq i)$$

III O

$$\lim_{\substack{m \rightarrow \infty \\ m \geq k+1}} P_i\left(\bigcup_{n=k+1}^m (X_n \in A_n) \mid X_k = i, X_{k-1} \neq i, \dots, X_1 \neq i\right)$$

III O

$$P_i\left(\bigcup_{n=k+1}^{\infty} (X_n \in A_n) \mid X_k = i, X_{k-1} \neq i, \dots, X_1 \neq i\right)$$

$$P_i\left(\bigcup_{n=1}^{\infty} X_n \in A_n\right)$$

$$P_i(X_{k+1} = x_{k+1}, \dots, X_m = x_m \mid X_k = i, X_{k-1} \neq i, \dots, X_1 \neq i) \\ \parallel \\ P_i(X_1 = x_1, \dots, X_{m-k} = x_m)$$

$$= P_i\left(1 + \sum_{n=k+1}^{\infty} 1(X_n = i) \geq m \mid X_k = i, X_{k-1} \neq i, \dots, X_1 \neq i\right)$$

$$= P_i\left(\sum_{n=k+1}^{\infty} 1(X_n = i) \geq m-1 \mid X_k = i, X_{k-1} \neq i, \dots, X_1 \neq i\right)$$

$$= P_i\left(\sum_{n=1}^{\infty} 1(X_n = i) \geq m-1\right)$$

$$= P_i(N(i) \geq m-1) - \textcircled{*X}$$

Exercise

$$\begin{aligned} \text{So } P_i(N(i) \geq m) &= \sum_{k \in \mathbb{N}^+} P_i(N(i) \geq m-1) P_i(T_i = k) \\ &= P_i(N(i) \geq m-1) P_i(T_i < \infty) \end{aligned} \quad \left(\because P_i(T_i < \infty) = \sum_{k \in \mathbb{N}^+} P_i(T_i = k) \right)$$

Iterating & inductively we get for all $m \geq 1$

$$P_i(N(i) \geq m) = (P_i(T_i < \infty))^m \quad (***)$$

$$\begin{aligned} P_i(N(i) = \infty) &= \lim_{m \rightarrow \infty} P_i(N(i) \geq m) \\ &= \lim_{m \rightarrow \infty} (P_i(T_i < \infty))^m \\ &= \begin{cases} 1 & \text{if } P_i(T_i < \infty) = 1 \\ 0 & \text{if } P_i(T_i < \infty) < 1 \end{cases} \end{aligned}$$

$i \in S$
recurrent
 $\Updownarrow$
 $P_i(T_i < \infty) = 1$

$$(N(i) = \infty) = \bigcap_{m=1}^{\infty} (N(i) \geq m)$$

D

Prop 1.6.2

$i \in S$ is recurrent $\Leftrightarrow P_i(T_i < \infty) = 1$

Remark:

(a) $P_i(N(i) = \infty) = \begin{cases} 1 & \text{if } P_i(T_i < \infty) = 1 \\ 0 & \text{otherwise} \end{cases}$

$P_i(N(i) = \infty) = 1 \Leftrightarrow E_i(N(i)) = \infty$

$\Uparrow$

$P_i(T_i < \infty) = 1$

$i \in S$ is recurrent
 $\Leftrightarrow$
 $E_i(N(i)) = \infty$ (X)

$P_i(N(i) < \infty) = 1 \Leftrightarrow P_i(T_i < \infty) < 1 \Leftrightarrow \sum_{m=1}^{\infty} (P_i(T_i < \infty))^m < \infty \Leftrightarrow \sum_{m=1}^{\infty} P_i(N(i) \geq m) < \infty$
 $\Downarrow$
 $E_i(N(i)) < \infty$

From (*)

$$i \in S \text{ is recurrent} \Leftrightarrow E_i(N(i)) = \infty$$

$$\Leftrightarrow E_i \left(\sum_{m=1}^{\infty} 1(X_m = i) \right) = \infty$$

$$E \left(\sum_{m=1}^n 1(X_m = i) \right) \overset{+}{=} \text{Monotone convergence}$$

$$\sum_{m=1}^n E(1(X_m = i))$$

$$\sum_{m=1}^{\infty} E_i(1(X_m = i)) = \infty$$

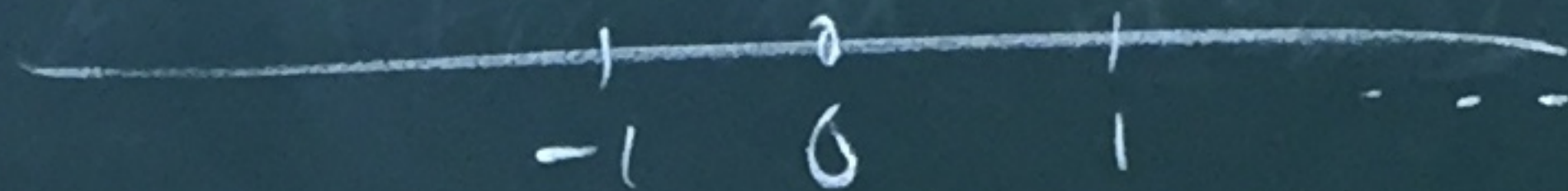
$$\Leftrightarrow \sum_{m=1}^{\infty} P_i(X_m = i) = \infty$$

$$\Leftrightarrow \sum_{m=1}^{\infty} \frac{m}{i} = \infty$$

$$i \in S \text{ is recurrent} \Leftrightarrow \sum_{m=0}^{\infty} p_{ii}^m = \infty$$

Example 1.6.3: $(X_n)_{n \geq 0}$ random walk on $\mathbb{Z}$, with natural
 • weights μ (i.e. $\mu_{i,i+1} = 1 = \mu_{i-1,i}$ & $\mu_{ij} = 0$ otherwise)

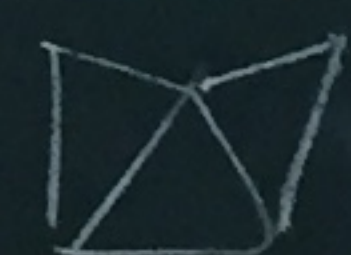
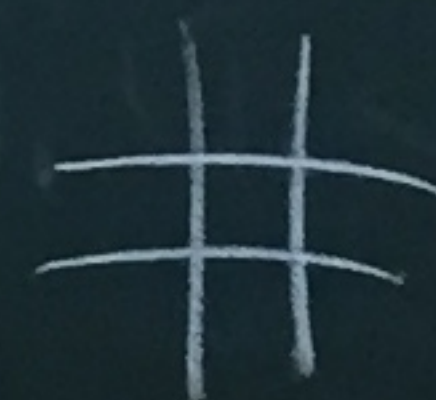
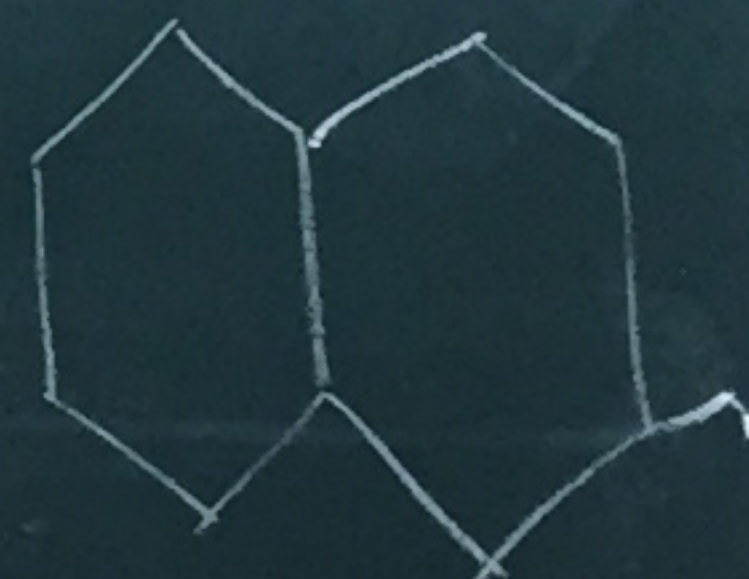
$$0 \in \mathbb{Z}$$



$$p_{00}^n = \begin{cases} 0 & n = 2k+1 \\ \binom{2k}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^k & n = 2k \end{cases}$$

$$\sum_{n=1}^{\infty} p_{00}^n = \sum_{k=1}^{\infty} 2^k \binom{2k}{k} \frac{1}{4^k}$$

Ex (Stirling's formula) $2^k \binom{2k}{k} \approx \frac{4^k}{\sqrt{k}}$



ch

$p_{00}^k \sim \frac{4^k}{\sqrt{k}} \frac{1}{4^k} = \frac{1}{\sqrt{k}}$ (large k)

$\left(\exists k_0, C_1, C_2 \text{ such that } C_2 \frac{4^k}{\sqrt{k}} \leq 2^k q_k \leq C_1 \frac{4^k}{\sqrt{k}} \quad \forall k \geq k_0 \right)$

$$\Rightarrow \sum_{k=1}^{\infty} p_{00}^k = \infty$$

change weights to

$$M_{i,i+1} = \alpha^i \quad 0 < \alpha < 1 \quad M_{i+1,i} = \alpha^i$$

$$M_{ij} = 0 \text{ otherwise.}$$

$$p_{i,i+1} = \frac{M_{i,i+1}}{M_i} = \frac{\alpha^i}{M_{i,i+1} + M_{i,i-1}} = \frac{\alpha^i}{\alpha^i + \alpha^{i-1}} = \frac{\alpha}{1+\alpha} = p$$

$$p_{i,i-1} = \dots$$

$$= \frac{1}{1+\alpha} = q$$

$$(p/q) = \alpha$$

Recall
Gambler's Ruin
 $p < q \quad h^{(i)}(i) = 1$
 $p > q \quad h^{(i)}(i) = (p/q)^i$

$$p_{00}^n = \begin{cases} 0 & n - \text{odd} \\ 2^k c_k (p)^k (q_V)^k & n = 2k \end{cases}$$

$$\sum_{n=0}^{\infty} p_{00}^n = \sum_{k=0}^{\infty} 2^k c_k (p)^k (q_V)^k \stackrel{(Ex)}{\sim} \sum_{k=0}^{\infty} \frac{(4pq_V)^k}{\sqrt{k}}$$

$$\alpha = 1 \quad (\Leftrightarrow) \quad 4pq_V = 1 \quad (\Leftrightarrow) \quad \sum_{n=0}^{\infty} p_{00}^n = \infty$$

$$0 \in S \cap \text{recurrent} \quad (\Leftrightarrow) \quad \alpha = 1$$

(a) $i \in S$ is recurrent & $i \rightarrow j$ for some $j \in S$

$\Rightarrow j$ is recurrent ✓

$$\text{e } P_i(T_j < \infty) = 1$$

$$P_j(T_i < \infty) = 1 \checkmark$$

$\Rightarrow \exists n_0 \geq 1$ such that

$$p_{ij}^{n_0} > 0 \text{ \& } p_{ij}^m = 0 \quad \forall m \leq n_0$$

$\Rightarrow \exists i_1, \dots, i_{n_0-1}$ such that $i_k \neq j \quad k \in \{1, \dots, n_0-1\}$
 & $i_k \neq i$

$$p_{ij} p_{i_1 i_2} \dots p_{i_{n_0-1} i} > 0$$

Suppose $P_j(T_i < \infty) < 1 \Rightarrow P_i(T_i = \infty) \geq p_{ii} \dots p_{i_{n_0-1} j} P_j(T_i = \infty) > 0$

$\Rightarrow P_i(T_i < \infty) < 1 \Rightarrow i \in S$ is not recurrent (contradiction)

$$(*) \quad \boxed{P_j(T_i < \infty) = 1 \quad \forall j \in S, \text{ such that } i \rightarrow j, \delta \neq \infty}$$

$\sum_{i=j}^{\infty}$

From $(*) \quad \exists n_1 \in \mathbb{N} \quad p_{ji}^{n_1} > 0$

We already know $\exists n_0 \in \mathbb{N} \quad p_{ij}^{n_0} > 0$

$$\Rightarrow \forall n > 1 \quad p_{ij}^{n_1 + n + n_0} \geq p_{ji}^{n_1} p_{ii}^n p_{ij}^{n_0}$$

Also

$\Rightarrow$

$$p_{jj}^{n+n_0}$$

$$\geq \begin{pmatrix} p_{ji}^{n_1} & p_{ij}^{n_0} \end{pmatrix} p_{ii}^n - (+)$$

$i \in S$ recurrent

$$\stackrel{(+)}{\Rightarrow} \sum_{n=1}^{\infty} p_{ii}^n = \infty$$

$(++)$

$$\Rightarrow \sum_{n=1}^{\infty} p_{ji}^n = \infty$$

$\stackrel{(+)}{\Rightarrow} j \in S$ is recurrent. (xxx)

Also note $j \rightarrow i$ by $(*)$, is recurrent $\Rightarrow P_i(T_j < \infty) = 1$ similarly $\square$