Recall :-

$\{X_n\}_{n\geq 0}$ M.C on state space S, transition matrix P

and initial distribution μ.

$A \subseteq S$ $\quad T^A = \inf\{n \geq 0 \mid X_n \in A\}$

$h^A : S \to [0,1] \qquad h^A(i) = \mathbb{P}_i(T^A < \infty)$

Thm 1.5.2 :- $A \subseteq S$ $\quad h^A$ is the minimal solution

st $\quad (1.5.3) \begin{cases} h^A(i) = 1 & \forall i \in A \\ h^A(i) = \sum_{j \in S} P_{ij} h^A(j) & \forall i \notin A \end{cases}$

Examp

$A = \{0$

Using
ideas
from re

Example 1.54 (Gambler's Ruin) :-

$(X_n)_{n\geq 0}$ M.C on $S = \{0\} \cup \mathbb{N}$.

$p_{00} = 1 \qquad p_{i,i+1} = p \quad \& \quad p_{i,i-1} = q \quad i \neq 0$

$A = \{0\}$ $\qquad h^A(i) = \mathbb{P}_i(T^A < \infty)$

- $h^A(0) = 1$

$h^A(i) = p\, h^A(i+1) + q\, h^A(i-1)$

$p < q \implies h^A(i) = 1 \quad \forall i \in S$

$p = q \implies h^A(i) = 1 \quad \forall i \in S$

$p > q \implies h^A(i) = (q/p)^i \quad \forall i \in S$

Using $a x_{n+1} + b x_n + c x_{n-1} = 0$
idea
from recurrence relation

Example 1.55 :- [Birth & Death chains]

$\{X_n\}_{n \geq 0}$ M.C on $S = \{0\} \cup \mathbb{N}$

$i \neq 0$

$p_{00} = 1$, $p_{i,i+1} = p_i$ and $p_{i,i-1} = 1 - p_i$ $\quad 0 < p_i < 1$.

$A = \{0\}$.

$$\begin{cases} h^A(0) = 1 \\ h^A(i) = p_i h^A(i+1) + q_i h^A(i-1) \end{cases} \quad [\text{By } (1.53)]$$

Another idea :- $u_i = h(i-1) - h(i)$

$\Rightarrow u_{i+1} = \left(q_i/p_i\right) u_i$ $\quad \overset{\text{(inductively)}}{\underset{\text{Iterate}}{\Rightarrow}}$ $\quad u_{i+1} = \left[\prod_{j=i}^{1} \left(q_i/p_i\right)\right] u_1$

$$V_{i+1} = \gamma_i u_1 \qquad \gamma_i = \prod_{j=1}^{i} \left(\frac{q_j}{p_j} \right)$$

$$\sum_{k=1}^{i} u_k = \hat{h}(0) - \hat{h}(i)$$

$$\hat{h}(i) = 1 - \left(\sum_{k=1}^{i} \gamma_{k-1} \right) u_1$$

$$\boxed{\hat{h}(i) = 1 - \left(\sum_{k=0}^{i-1} \gamma_k \right) u_1} \quad (*)$$

- $\sum_{k=0}^{\infty} \gamma_k = \infty \implies u_1 = 0 \implies 0 = h^A(0) - h^A(1) \implies h^A(1) = 1$

$\implies h^A(i) = 1$ from $(*)$ for all $i \in S$.

- $\sum_{k=0}^{\infty} \gamma_k < \infty \implies$ there are u_1 such that $0 \le h^A(i) \le 1 \ \forall i \in S$.

Minimal Solution $:\equiv$ $u_1 = \dfrac{1}{\sum_{k=0}^{\infty} \gamma_k}$

$$(\text{Thm } 1.5.2) \implies \boxed{h^A(i) = \frac{\sum_{k=i}^{\infty} \gamma_k}{\sum_{k=0}^{\infty} \gamma_k}}$$

$\square$

Mean hitting times

$\{X_n\}_{n\geq 0}$ M.C.C on state space S & Transition Matrix P & initial distribution μ

$A \subseteq S \qquad T^A = \inf\{n \geq 0 \mid X_n \in A\}$

$$K^A(i) = \mathbb{E}_i[T^A] = \begin{cases} \infty & \text{if } \mathbb{P}_i(T^A = \infty) > 0 \\ \sum_{k=1}^{\infty} k\, \mathbb{P}_i(T^A = k), & \text{otherwise} \end{cases}$$

Theorem 1.5.5 $A \subseteq S$ $\mathbb{P}_i(T^A < \infty) = 1$ $\forall i \in S$

$\cdot$ $\mathbb{E}_i(T^A) < \infty$ for all $i \in S$

$\{k^A(i) : i \in S\}$ - vector of mean hitting times

is the minimal non-negative solution to the

System of linear equations

$$(1.5.6)\begin{cases} k^A(i) = 0 & \text{if } i \in A \\ k^A(i) = 1 + \sum_{j \notin A} p_{ij} k^A(j) & i \notin A \end{cases}$$

Proof

$\cdot$ k^A

term by term

" Series

Calculat

Proof:

- $i \in A$ $\quad \mathbb{P}_i(T^A = 0) = 1 \Rightarrow \mathbb{E}_i[T^A] = 0 \Rightarrow k^A(i) = 0$

- $$k^A(i) = \sum_{k=0}^{\infty} k\,\mathbb{P}_i(T^A = k)$$

Proof Thm 1.5.2

term by term "Series" calculation

$$= \mathbb{P}_i(T^A = 1) + \sum_{k=2}^{\infty} k \left(\sum_{j \in S \setminus A} p_{ij}\, \mathbb{P}_j(T^A = k-1) \right)$$

$$= \mathbb{P}_i(X_1 \in A) + \sum_{j \in S \setminus A} p_{ij} \left(\sum_{k=2}^{\infty} k\, \mathbb{P}_j(T^A = k-1) \right)$$

$$= \sum_{j \in A} p_{ij} + \sum_{j \in S \setminus A} p_{ij} \left(\sum_{k=1}^{\infty} (k+1)\, \mathbb{P}_j(T^A = k) \right)$$

$$= \sum_{j \in A} p_{ij}$$

$\sum_{k=1}^{\infty} k\,\mathbb{P}_j(T^A = k) = \mathbb{E}_j(T^A)$

$$= \sum_{j \in A} p_{ij}$$

$(\mathbb{P}_j(T^A < \infty) = 1)$

$\sum_{j \in S}$

$= 1 +$

$$= \sum_{j \in A} p_{ij} + \sum_{j \in S/A} p_{ij} \left(\sum_{k=1}^{\infty} \mathbb{P}_j (T^A = k) \right) + \sum_{j \in S/A} p_{ij} \, \mathbb{E}_j (T^A)$$

$$\sum_{k=1}^{\infty} k \, \mathbb{P}_j (T^A = k) = \mathbb{E}_j (T^A)$$

$$= \sum_{j \in A} p_{ij} + \sum_{j \in S/A} p_{ij} \, \mathbb{P}_j (T^A < \infty) + \sum_{j \in S/A} p_{ij} \, \mathbb{E}_j (T^A)$$

$(\mathbb{P}_j (T^A < \infty) = \underline{1})$

$$\sum_{j \in S} p_{ij} + \sum_{j \in S/A} p_{ij} \, \mathbb{E}_j (T^A)$$

$$\boxed{\begin{array}{l} \text{Assumed} \\ k^A(j) < \infty \ \forall j \in S \end{array}}$$

$$= 1 + \sum_{j \in S/A} p_{ij} \, k^A(j)$$

$\{y_i : i \in S\}$ be a solution to (1.5.6)

$$y_i = 0 \quad \forall i \in A \quad , \quad k^A(i) = 0 \quad \forall i \in A$$

$$\therefore \; y_i = k^A(i) \quad \forall i \in A$$

$i \notin A$
$$y_i = 1 + \sum_{j \notin A} p_{ij} y_j$$

(Iterate)
$$= 1 + \sum_{j \notin A} p_{ij} \left(1 + \sum_{k \notin A} p_{jk} y_k \right)$$

$$= 1 + \sum_{j \notin A} p_{ij} + \sum_{j \notin A} \sum_{k \notin A} p_{ij} p_{jk} y_k$$

$$= \mathbb{P}_i(T^A \geq 1) + \mathbb{P}_i(T^A \geq 2) + \sum_{j \notin A} \sum_{k \notin A} p_{ij} p_{jk} y_k$$

<u>Inductively</u>, repeat 'n' times.

$$y_i = \sum_{k=1}^{n} \mathbb{P}_i(T^A \geq k) + \sum_{\substack{j \notin A \\ i_k \notin A \\ k \leq n-1}} p_{ij} p_{ji_1} \cdots p_{i_{n-2} i_{n-1}} \underbrace{y_{i_{n-1}}}_{\geq 0}$$

$(\text{as } y_i \geq 0)$

$$\boxed{\begin{aligned} &\mathbb{P}_i(T^A = 1) = \sum_{j \in A} p_{ij} \\ &\mathbb{P}_i(T^A = 2) = \sum_{j \notin A} p_{ij} \Big(\sum_{k \in A} p_{kj} \Big) \\ &\mathbb{P}_i(T^A \geq 1) = 1 \\ &\mathbb{P}_i(T^A \geq 2) = 1 - \mathbb{P}_i(T^A = 1) = 1 - \sum_{j \in A} p_{ij} = \sum_{j \notin A} p_{ij} \end{aligned}}$$

$$\Rightarrow \quad y_i \geq \sum_{k=1}^{n} \mathbb{P}_i(T^A \geq k) \quad \forall n \geq 1.$$

$$\Big\downarrow \text{ as } n \to \infty$$

$$\therefore \quad y_i \geq \mathbb{E}_i(T^A) = k^A(i) \qquad \square$$