Recall:

$(X_n)_{n\geq 0}$ Markov chain on state space S with transition matrix P with initial distribution μ.

- π - a probability on S is a stationary distribution for $(X_n)_{n\geq 0}$ if

$$\pi(j) = \sum_{i\in S} \pi(i)\, p_{ij} \qquad \forall j \in S.$$

- $X_0 \overset{d}{=} \pi$ then $X_n \overset{d}{=} \pi$

stationary $\Uparrow$ $\Downarrow$? steady state

- P - doubly stochastic, $|S|<\infty$ then $\underline{\text{a}}$ stationary distribution is given by $\pi(j) = \frac{1}{|S|}$ $j\in S$

$\boxed{\text{Is it unique?}}$

$\longleftarrow$ Suppose $\exists$ $|S|<\infty$ $p_{ij}^{n} \to \pi(j)$ $\forall j\in S$ then π is a stationary distribution

Theorem

[if

Proof:

Fix $i \in S$.

Theorem 1.9.8 Let $N_n(i) = \sum_{k=1}^{n} \mathbb{1}(X_k = i)$. $(X_n)_{n\geq 0}$ is irreducible.

Then $\frac{N_n(i)}{n} \longrightarrow \frac{1}{E_i(T_i)}$ with probability one.

$$\left[\begin{array}{ll} \text{if } E_i(T_i) < \infty, & \mathbb{P}\left(\lim_{n\to\infty} \frac{N_n(i)}{n} = \frac{1}{E_i(T_i)}\right) = 1 \\ E_i(T_i) = \infty & \mathbb{P}\left(\lim_{n\to\infty} \frac{N_n(i)}{n} = 0\right) = 1 \end{array}\right]$$

Proof: (I) $i \in S$, i is transient

- $N_n(i) \leq \sum_{k=1}^{\infty} \mathbb{1}(X_k = i) = N(i)$
- $\mathbb{P}_i(N(i) < \infty) = 1$ $\quad \mathbb{P}_i(T_i = \infty) > 0$

$\Rightarrow E_i(T_i) = \infty$ & $\mathbb{P}\left(\lim_{n\to\infty} \frac{N(i)}{n} = 0\right) = 1$

As $N_n(i) \leq N(i)$

$1 \geq \mathbb{P}\left(\lim_{n\to\infty} \frac{N_n(i)}{n} = 0\right)$

$\geq \mathbb{P}\left(\lim_{n\to\infty} \frac{N(i)}{n} = 0\right) = 1$

(I) $i \in S$

let

Fact:

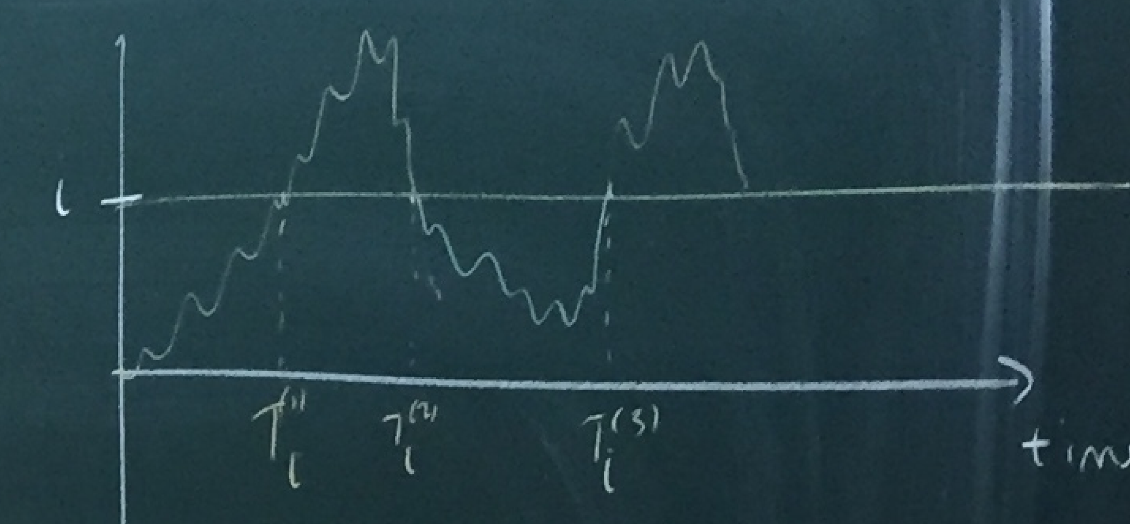
SLLN

$\therefore \quad \frac{N_n(i)}{n} \longrightarrow \frac{1}{E_i(T_i)}$ with probability one.

II $\quad i \in S \quad i$ is recurrent & $E_i(T_i) < \infty$.

let $\quad T_i^{(r)} = \inf\{k \geq 1 \mid N_k(i) = r\} = \{ \; T_i^{(r)} = \inf\{k > T_i^{(r-1)} \mid X_k = i\}$

$$S_i^{(r)} = T_i^{(r)} - T_i^{(r-1)}$$



$T_i^{(0)} = 0$

Fact:- $\{S_i^{(r)}\}_{r \geq 1}$ are i.i.d &

$$r \geq 1 \quad \mathbb{P}(S_i^{(r)} = n \mid T_i^{(r-1)} < \infty) = \mathbb{P}_i(T_i = n)$$

SLLN will imply $\quad \frac{\sum_{r=1}^{n} S_i^{(r)}}{n} \longrightarrow E_i[T_i]$ as $n \to \infty$ with probability one.

$\left(E[S_i^{(1)}] = E_i[T_i] \Leftarrow \mathbb{P}(S_i^{(1)} = n) = \mathbb{P}_i(T_i = n) \right)$

As $\dfrac{\sum_{r=1}^{n} S_i^{(r)}}{n} = \dfrac{T_i^{(n)}}{n} \Rightarrow \dfrac{T_i^{(n)}}{n} \xrightarrow[n\to\infty]{} E_i(T_i)$ w.p. 1

$A = \{ \omega \in \Omega \mid \dfrac{T_i^{(n)}(\omega)}{n} \xrightarrow[n\to\infty]{} E_i(T_i) \}$ $\quad P(A) = 1$

$B = \{ \omega \in \Omega \mid N_n(i)(\omega) \xrightarrow[n\to\infty]{} \infty \}$ $\quad P(B) = 1$

$\left.\begin{array}{l} P(A)=1 \\ P(B)=1 \end{array}\right\} \mathbb{P}(A \cap B) = 1$

$C = A \cap B$

$\omega \in C \Rightarrow \dfrac{T_i^{N_n(i)}}{N_n(i)} \longrightarrow E_i(T_i)$

[Fact: $a_n \to a$ as $n \to \infty$, $b_n \to \infty$ as $n \to \infty \Rightarrow a_{b_n} \to a$ as $n \to \infty$]

Observe

$\Bigl(\omega \in C$

$N_n(i) > 0$ eventually $\Bigr)$

$\omega \in C$

As $(T$

Observe:- $T_i^{N_n(i)} \le n < T_i^{N_n(i)+1}$ " $T_i^{(\cdot)} \equiv N_\cdot(i)^{-1}$ "

$$\left(\begin{array}{l} w \in C \\ N_n(i) > 0 \\ \text{eventually} \end{array} \right) \implies \frac{T_i^{N_n(i)}}{N_n(i)} \le \frac{n}{N_n(i)} \le \frac{T_i^{N_n(i)+1}}{N_n(i)+1} \frac{N_n(i)+1}{N_n(i)}$$ - for large enough n

$w \in C$, $n \to \infty$ $\quad E_i(T_i) = \lim_{n\to\infty} \frac{n}{N_n(i)}$, by Squeeze Theorem.

As $\mathbb{P}(C) = 1$ & $E_i(T_i) < \infty$

$$\implies \mathbb{P}\left(\lim_{n\to\infty} \frac{N_n(i)}{n} = \frac{1}{E_i(T_i)} \right)$$

III. $i \in S$ is recurrent, $E_i(T_i) = \infty$.

Fact

- $T_i^{(r)} = \inf\{k \geq 1 \mid N_k(i) = r\}$ $\quad S_i^{(r)} = T_i^{(r)} - T_i^{(r-1)}$ iid &

  $\mathbb{P}(T_i^{(r)} < \infty) = 1$ & $\mathbb{P}(S_i^{(r)} = n \mid T_i^{(1)} < \infty) = \mathbb{P}_i(T_i = n)$

- $E[S_i^{(1)}] = \infty$. Version of SLLN $\quad \dfrac{\sum_{r=1}^{n} S_i^{(r)}}{n} \longrightarrow \infty$ w.p. 1

  $S_i^{(r)} \geq 0 \ \forall r \geq 1$

Same proof as II applies

- $A = \left\{\dfrac{T_i^{(n)}}{n} \to \infty\right\}$ $\quad B = \{N_n(i) \to \infty\}$ $\quad C = A \cap B$, $\quad \dfrac{T^{N_n(i)}}{N_n(i)} \to \infty$ as $n \to \infty$ on C

- $\infty = \lim_{n \to \infty} \dfrac{n}{N_n(i)}$ on C, $\quad P(C) = 1 \Rightarrow \mathbb{P}\left(\lim_{n \to \infty} \dfrac{N_n(i)}{n} = 0\right) = 1$. $\square$

  $0 = \lim_{n \to \infty} N_n(i)/n$ on C

Theorem 1.9.7 Let $(X_n)_{n\geq 0}$ be a Markov chain on state space S with transition matrix P. Suppose $|S|<\infty$ & P is irreducible. Then $(X_n)_{n\geq 0}$ has a stationary distribution.

Proof:

Directed Spanning tree rooted at $i \in S$

$g \equiv$ graph $\quad S \equiv$ vertices of g

(a) $E_g \subseteq \{(i,j) \mid i\in S, j \in S\}$ ordered — Directed Edges

(b) g - has no cycles ie $\nexists\, n \geq 1$, $j \in S$ such that $x_0 = j, x_1, x_2, \ldots x_n = j$ $\quad (x_i, x_{i+1}) \in E_g$

(c) $(i,l) \notin E_g \;\forall\, l \in S$, $\quad j \neq i \;\exists$ unique k $(j,k) \in E_g$

Example

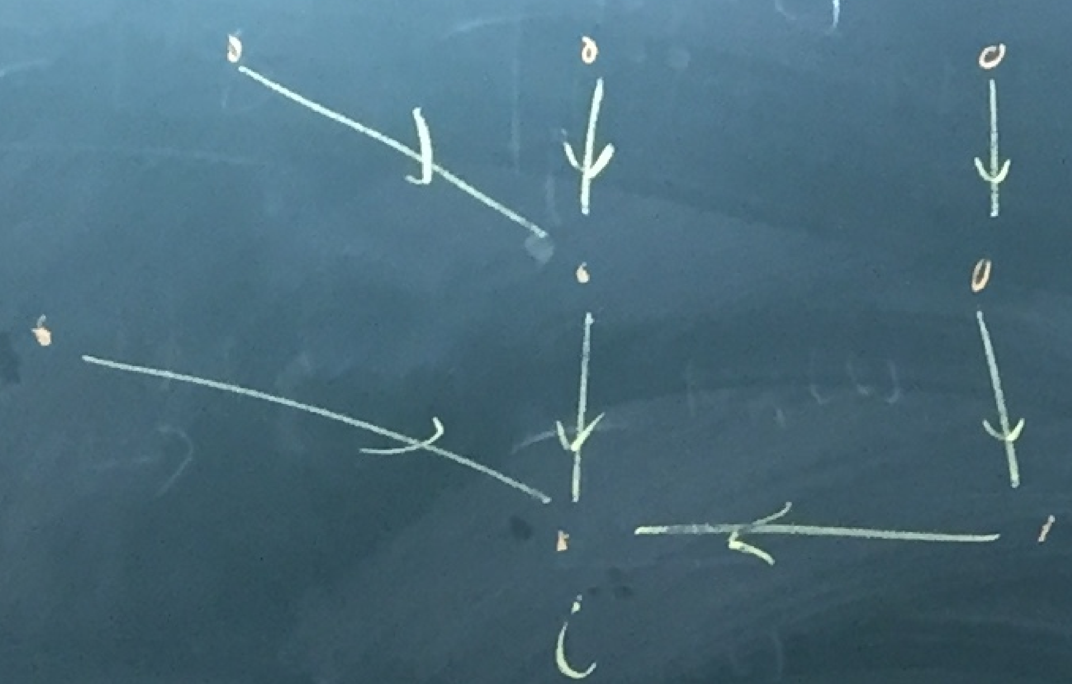
let $\pi_0(i) =$

(Irreducibility)

$0 < c \triangleq \sum_{i\in S} \pi_0(i) <$

Now π is

Example



$$G(i) \equiv \left\{ g \;\middle|\; g \text{ is directed ``spanning tree rooted at'' } i \in S \right\}$$

let $$\pi_0(i) = \sum_{g \in G(i)} \prod_{(l,k) \in E_g} p_{lk}$$

(Irreducibility)

$$0 < c \triangleq \sum_{i \in S} \pi_0(i) < \infty \implies \pi(i) = \frac{\pi_0(i)}{c} \text{ is a Probability on } S.$$

Now π is stationary for $\{X_n\}_{n \geq 0}$ iff

$$\pi(j) = \sum_{i \in S} \pi(i)\, p_{ij} \quad \forall j \in S. \quad \text{Fix } j \in S.$$

III. Equivalently

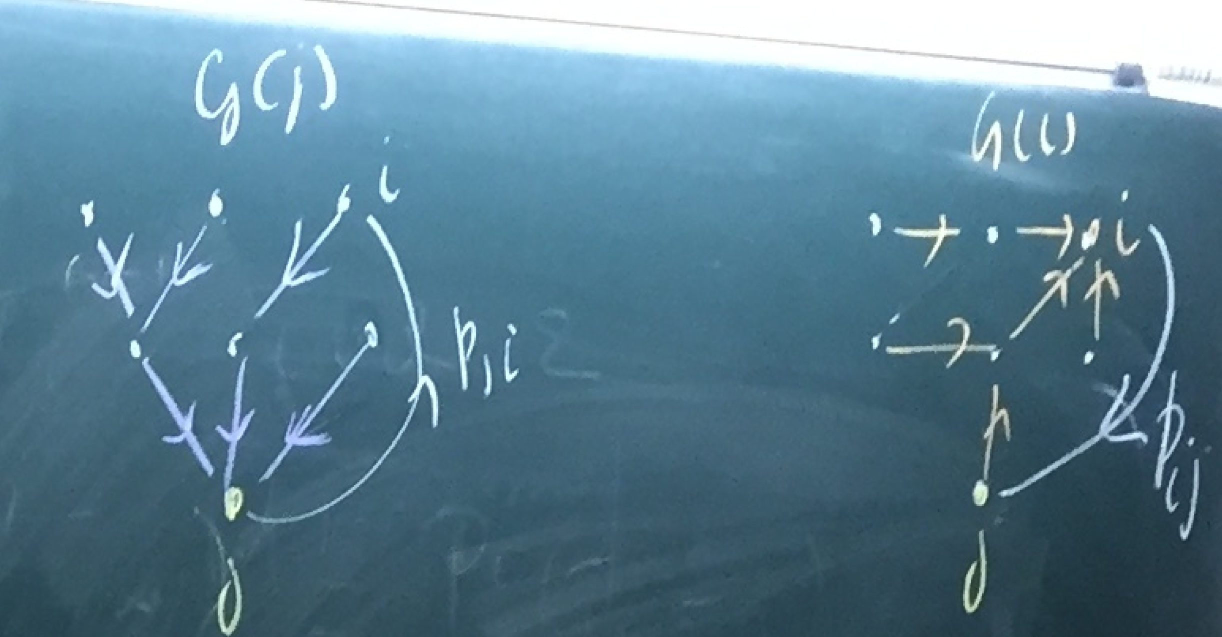
It is Enough to show: $\pi_0(j) = \sum_{i\in S} \pi_0(i)\, p_{ij}$

$$\iff \pi_0(j)(1-p_{jj}) = \sum_{\substack{i\in S\\ j\neq i}} \pi_0(i)\, p_{ij}$$

$$\iff \pi_0(j)\Big(\sum_{\substack{i\in S\\ j\neq i}} p_{ji}\Big) = \sum_{\substack{i\in S\\ j\neq i}} \pi_0(i)\, p_{ij}$$

$$\iff \Big(\sum_{g\in G(j)} \prod_{(l,k)\in E_g} p_{lk}\Big)\Big(\sum_{\substack{i\in S\\ j\neq i}} p_{ji}\Big) = \sum_{\substack{i\in S\\ j\neq i}}\Big(\sum_{g\in G(i)} \prod_{(l,k)\in E_g} p_{lk}\Big) p_{ij}$$

$$\iff \sum_{g\in G(j)} \sum_{\substack{i\in S\\ j\neq i}} \prod_{(l,k)\in E_g} p_{lk}\, p_{ji} = \sum_{i\in S,\, j\neq i} \sum_{g\in G(i)} \prod_{(l,k)\in E_g} p_{lk}\, p_{ij} \quad (*)$$



$L \equiv$ set of all graphs, S-vertices

- $\forall\, m\in S\ \exists$ unique $n\in S$ $(m\neq n)$ $(m,n)\in S$
- $\exists$ one cycle at j

$$\sum_{g\in L} \prod_{(l,k)\in E_g} p_{lk} = LHS_{(*)} = RHS_{(*)}$$