

Random Walks on graphs

$$(P, \mu) \quad T = (V, E) \quad V - \text{countable}$$
$$E \subseteq P_2(V) = \{ \{x, y\} \mid x, y \in V \}$$

$$x \sim y \quad \text{if } \{x, y\} \in E$$

path $\gamma: x = x_0, \dots, x_n = y$ from x to y , $|\gamma| = n$
if $x_i \sim x_{i+1}$

path is self-avoiding if $x_i \neq x_j$ for all $i \neq j$

$d(x, y) \equiv$ length of shortest path from x to y .
 $d(x, x) = 0$. [convention]

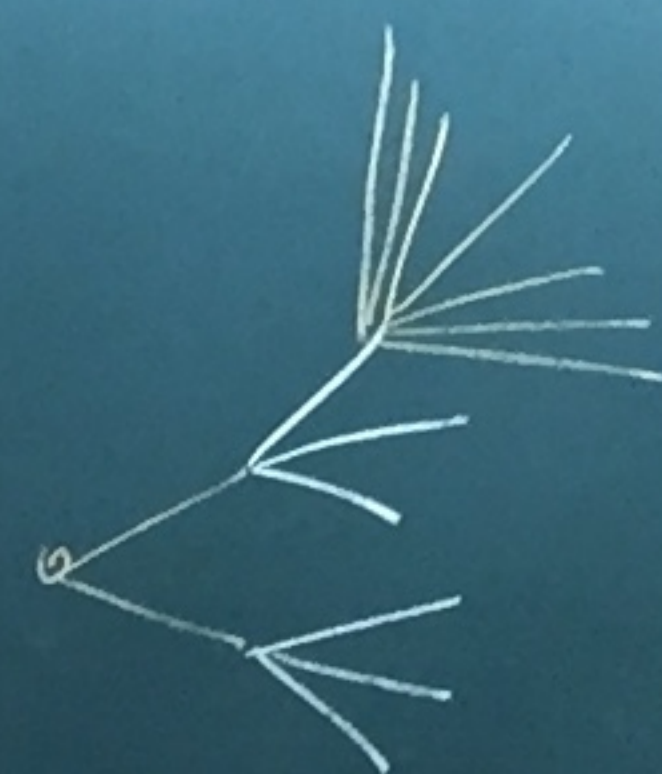
• Γ -connected

• Γ -locally finite

$$N(x) = |\{y \in V \mid y \sim x\}|$$

$$N(x) < \infty \text{ for all } x \in V.$$

$$\Gamma\text{-Bounded geometry} \quad \sup_{x \in V} N(x) < \infty$$



$$\mu: V \times V \rightarrow \mathbb{R}_+$$

(i) $\mu_{xy} = \mu_{yx}$

(ii) $\mu_{xy} \geq 0$ for all $x, y \in V$

(iii) $x \neq y \quad \mu_{xy} > 0 \Leftrightarrow x \sim y$

Natural weights

$$\mu_{xy} = \begin{cases} 1 & y \sim x \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_x = \sum_{y \sim x} \mu_{xy}$$

Extend μ to a countable additive set function:

$$\mu(A) = \sum_{x \in A} \mu_x \quad A \subseteq V$$

$\Rightarrow A_1, \dots, A_n, \dots$ countable collection

$$A_i \cap A_j = \emptyset$$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

$$B(x, n) = \{y \in V \mid d(x, y) \leq n\}$$

(2)

ditte set

$$A \subseteq V$$

etum

Ai)

$$y \in n^y$$

lemma : let (T, μ) have controlled weights, i.e. $\exists C > 0$ such $\frac{\mu_{xy}}{\mu_x} > \frac{1}{C}$ whenever $x \sim y$

(1) T has Bounded Geometry

$$(2) |B(x, n)| \leq 2C^n$$

$$(3) \mu(B(x, n)) \leq 2C^n \mu_x$$

Proof

$$(1) N(x) = \sum_{y \in V} 1_{(y \sim x)}$$

$$= C \sum_{y \in V} \frac{1}{C} 1_{(y \sim x)}$$

(2) & (3)

E_x

$$\leq C \sum_{y \in V} \frac{\mu_{xy}}{\mu_x} 1_{(y \sim x)}$$

$$= C \sum_{y \sim x} \mu_{xy} = C \square$$

$$V = \mathbb{Z}^d, E = \{ \{x, y\} \mid |x - y| = 1 \}, N(x) = 2d.$$

$V = \mathbb{T}^d$ - degree 'd' tree.

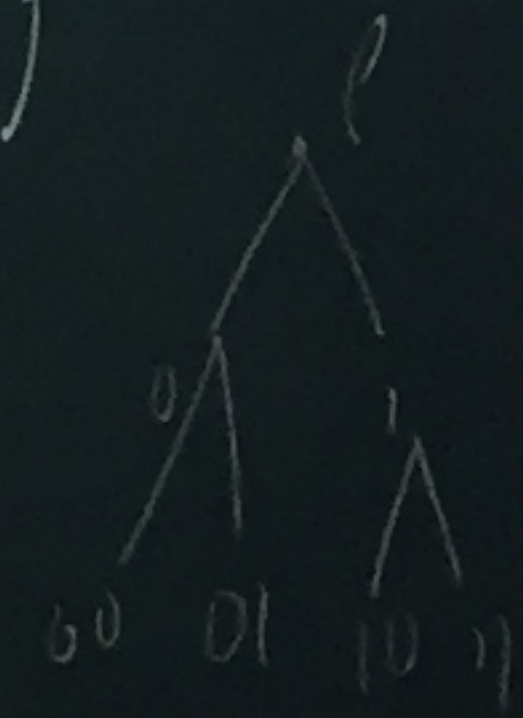
Binary tree $B_0 = \{p\}, B_n = \{0, 1\}^n$

$$V = B_0 \cup \bigcup_{n=1}^{\infty} B_n$$

$$x \in V \setminus \{p\} \Rightarrow x = (x_1, \dots, x_n) \text{ for some } n \geq 1$$

$$(parent of x) \quad d(x) = (x_1, \dots, x_{n-1})$$

$$E = \{ \{x, d(x)\} \mid x \in V \setminus \{p\} \} \cup \{ \{p, 0\}, \{p, 1\} \}$$



Canopy tree

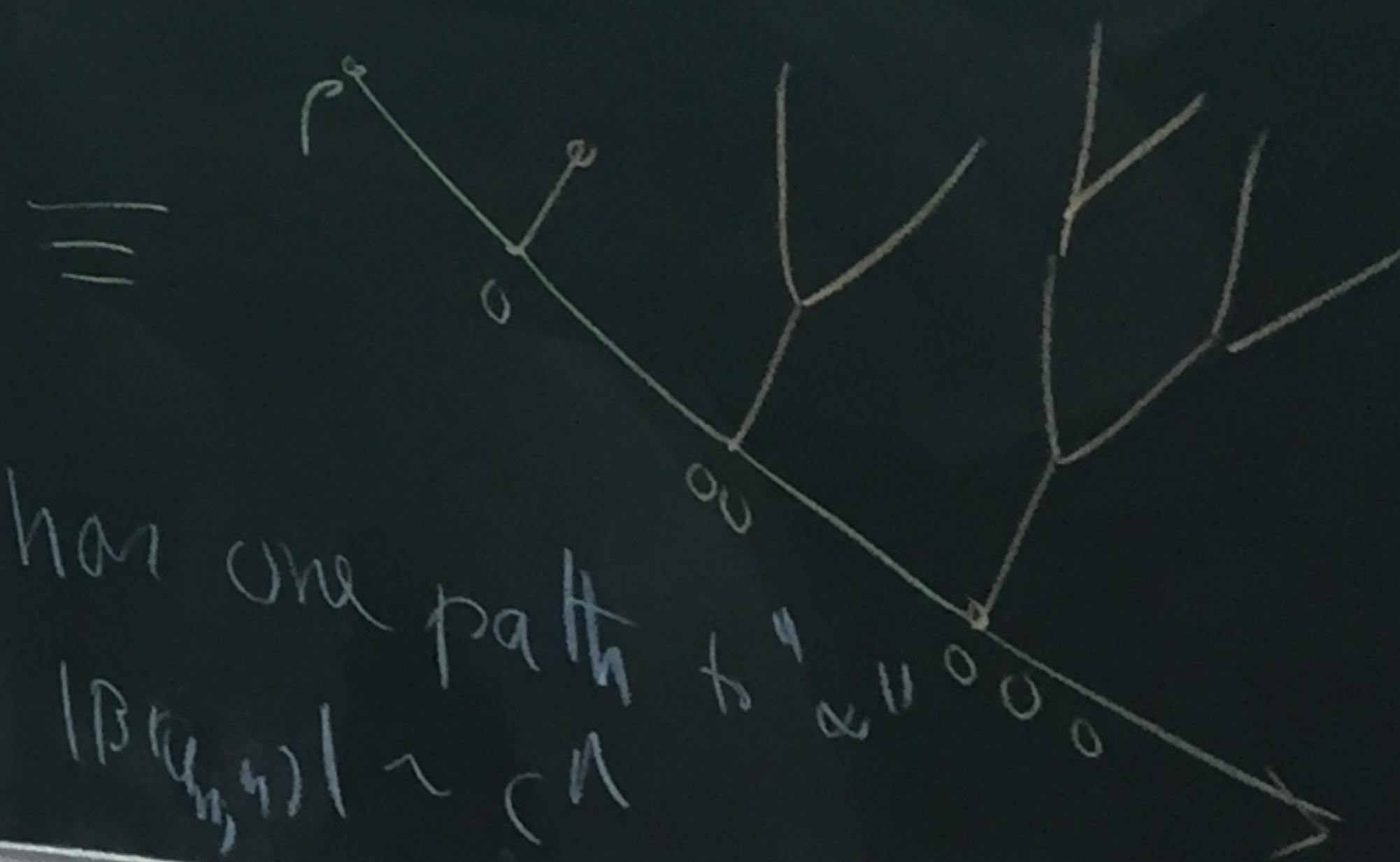
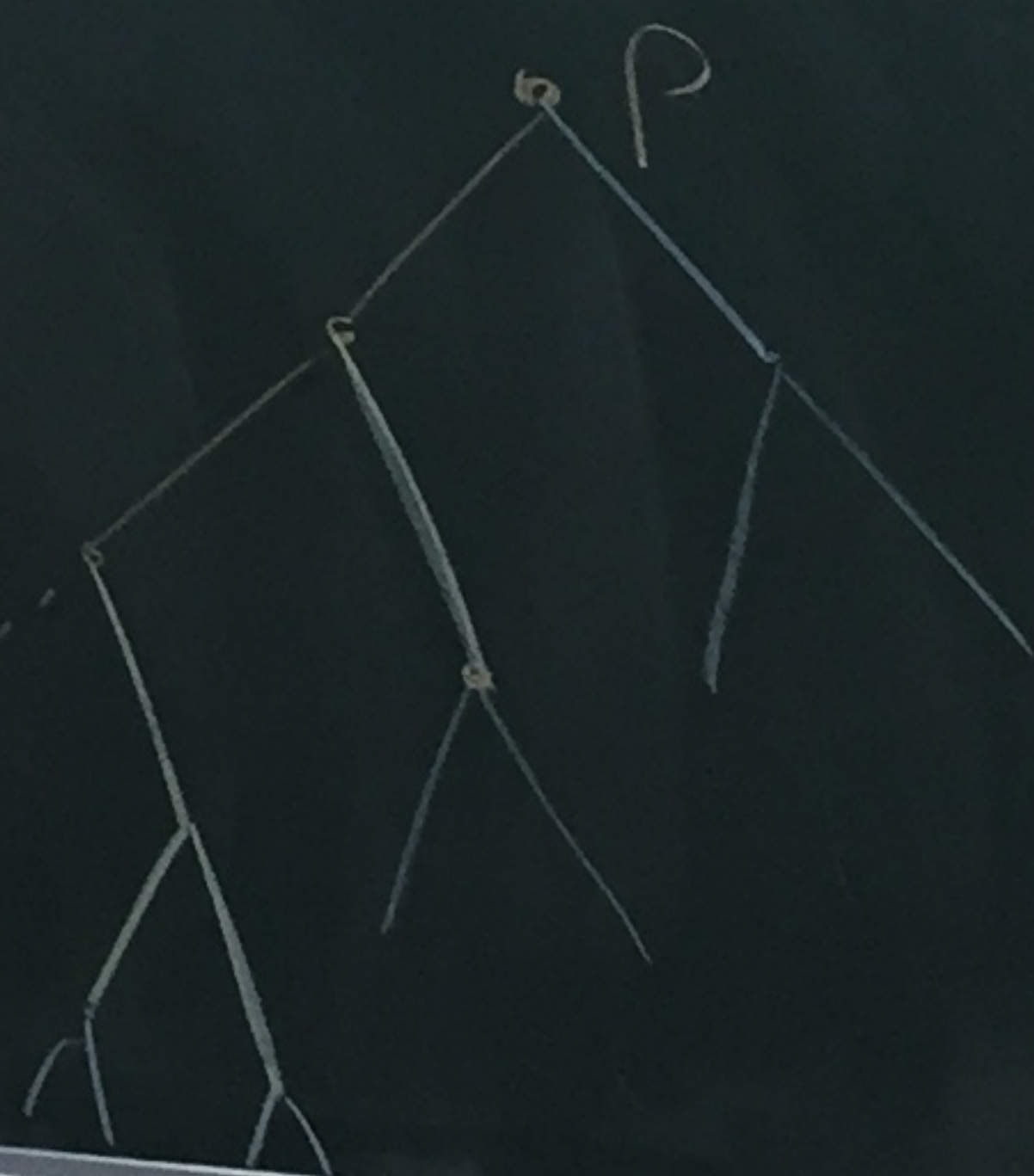
$U \subseteq V$, $U = \{p\} \cup \{x \in V \mid x = (x_1, \dots, x_n) \text{ s.t. } x_i = 0 \text{ for some } i \geq 1\}$
 ↳ Binary tree

$f: V \rightarrow V$

$y \in U$ where y is the closest point to x in U . then set $y = f(x)$

$V \supseteq V_{\text{canopy}} = \{x \in V \mid d(x, f(x)) \leq d(p, f(x))\}$

$E_{\text{canopy}} \subseteq E$ $x, y \in V_{\text{canopy}}$ $x \sim y$ in E_{canopy} $(\Leftrightarrow) x \sim y$ in E



- has one path to all nodes
 - $|B(p, r)| \sim n$

TR

(T1)

(T2)

(T3)

(T4)

(T5)

Random Walk on (P, μ) $(X_n)_{n \geq 0}$ - M.C. on V

$$T_A = \min\{n \geq 1 \mid X_n \in A\}, T^A = \min\{n \geq 0 \mid X_n \in A\}$$

transition matrix

$$p_{xy} = \frac{\mu_{xy}}{\mu_x}$$

P - locally finite & connected

Question

Kesten

TRANSIENT - Equivalent definitions

$$(T1) \quad \exists x \in V \quad P^x(T_x < \infty) < 1$$

$$(T2) \quad \forall x \in V \quad P^x(T_x < \infty) < 1$$

$$(T3) \quad \forall x \in V \quad \sum_{n=0}^{\infty} p_{xx}^n < \infty$$

$$(T4) \quad \forall x, y \in V \quad x \neq y \quad \text{either } P^x(T_y < \infty) < 1 \\ \text{or } P^y(T_x < \infty) < 1$$

$$(T5) \quad P^x\left(\sum_{n=0}^{\infty} 1_{\{X_n=y\}} < \infty\right) = 1 \quad \forall x, y \in V$$

Recurrent - Equivalent definitions

$$(R1) \quad \exists x \in V \quad P^x(T_x < \infty) = 1$$

$$(R2) \quad \forall x \in V \quad P^x(T_x < \infty) = 1$$

$$(R3) \quad \forall x \in V \quad \sum_{n=0}^{\infty} p_{xx}^n = \infty$$

$$(R4) \quad \forall x, y \in V \quad x \neq y \quad P^x(T_y < \infty) = 1 \\ P^y(T_x < \infty) = 1$$

$$(R5) \quad P^x\left(\sum_{n=0}^{\infty} 1_{\{X_n=y\}} = \infty\right) = 1 \quad \forall x, y \in V$$

Question: (P, μ) is it recurrent or transient?

Kesten Problem:

G - group - finitely generated

$\Lambda = \begin{Bmatrix} a_1, \dots, a_n \\ a_1^{-1}, \dots, a_n^{-1} \end{Bmatrix}$ as generating set

$V = G$ $\{a_i, h\} \in E$ if $g^{-1}h \in \Lambda$, $P \equiv$ Cayley graph

Question: Is (P, μ) recurrent or transient?

$\Gamma = (V, E)$
Theorem: (P, μ) is a weighted graph

$$\mu(v) = \sum_{x \in v} \mu_x < \infty$$

then the (P, μ) is recurrent

Known Techniques

$\mathbb{Z}$ - recurrent

$\mathbb{Z}^2$ - recurrent

$\mathbb{Z}^3$ - transient

$$P_{00}^{(2n)} = \dots \text{ (combinatorial expression) }$$

[Polya]

$$\xi_1 + \dots + \xi_n = X_n$$

"Fourier transform" $\leftarrow$

Not to

①

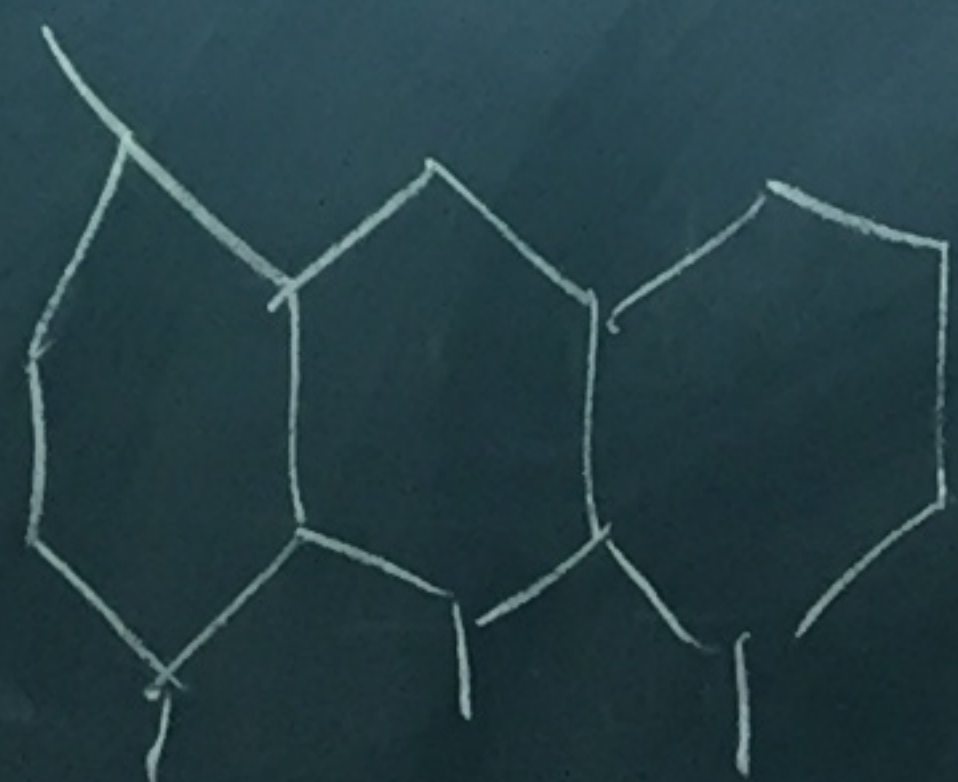
②

③

Q. "Technique"

Not robust

①



②

$\mathbb{Z}^d$ add "X"

③

$\mathbb{Z}^2$

some other weights

$\mu_{xy} \in [c_1, c_2]$

Q. "Techniques" that are stable under perturbation of the graph / weights?

Function Spaces

$$C(V) = \{f: V \rightarrow \mathbb{R}\}$$

$$f \in C(V)$$

$$e = \{u, v\} \quad \nabla f(e) = |f(u) - f(v)|$$

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} (f(y) - f(x))$$

- Graph Laplacian

$$\boxed{f'' = \Delta}$$

$$Df(x) = \sum_{y \in V} \frac{M_{xy}}{\mu_x} (f(y) - f(x)) = \sum_{y \in V} p_{xy} f(y) - f(x) \\ = E[f(X_1) | X_0 = x] - f(x)$$

Define $Pf(x) = \sum_{y \in V} p_{xy} f(y) = E[f(X_1) | X_0 = x]$

$$\boxed{D = P - I}$$

Dirichlet or Energy form

$f, g \in C(V)$, define a quadratic form

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_{x \in V} \sum_{y \in V} \mu_{xy} (f(y) - f(x))(g(y) - g(x))$$

where the sum converges absolutely.

Thm - [Discrete Gauss-Green]

$$f, g \in C(V) \quad \sum_{x \in V} \sum_{y \in V} |g(x)| |f(y) - f(x)| \mu_{xy} < \infty \quad (*)$$

$$\text{then } \mathcal{E}(f, g) = - \sum_{x \in V} \Delta f(x) g(x) \mu_x$$

$$\frac{1}{2} \int f \nabla g dx = \frac{1}{2} \int \nabla f \cdot \nabla g dx$$

Proof

Proof: $\varepsilon(f, g) = \frac{1}{2} \sum_{x \in V} \sum_{y \in V} \mu_{xy} (f(y) - f(x)) (g(y) - g(x))$

$\stackrel{(*)}{=} \frac{1}{2} \sum_{x \in V} \sum_{y \in V} \mu_{xy} (f(y) - f(x)) g(y) - \frac{1}{2} \sum_{x \in V} \sum_{y \in V} \mu_{xy} (f(y) - f(x)) g(x)$

$\mu_{xy} = \mu_{yx}$

$= - \sum_{x \in V} \sum_{y \in V} \frac{\mu_{xy} (f(y) - f(x))}{\mu_x} g(x) \mu_x$

$= - \sum_{x \in V} \Delta f(x) g(x) \mu_x \quad \square$

Proof of Theorem

1) $g(x)$

$$z \in V, \quad \phi(x) = P^x(T_z = \infty)$$

$$\phi(z) = 0 \quad (\text{Markov})$$

$$\text{Now } \phi(x) = P^x(T_z = \infty) \stackrel{\text{Markov}}{=} \sum_{y \in V} P^x(X_1 = y) P^y(T_z = \infty)$$

$$= \sum_{y \in V} p_{xy} \phi(y)$$

$$\Rightarrow \phi(x) = P\phi(x)$$

$$\Rightarrow (P - I)\phi(x) = 0 \quad \Rightarrow \Delta\phi(x) = 0$$

$$\mu(V) < \infty \Rightarrow \sum_{x \in V} |\phi(x)|^2 \mu_x \leq \frac{1}{\sum_{x \in V} \mu_x} = \frac{1}{\mu(V)} < \infty$$

$$[Ex] \Rightarrow \sum_{x \in V} \sum_{y \in V} |\phi(y) - \phi(x)| |\phi(x)| \mu_{xy} < \infty \quad (0 \leq \phi(x) \leq 1)$$

Apply Gauss-Green

$$\mathcal{E}(\phi, \phi) = - \sum_{x \in V} \Delta \phi(x) \phi(x) \mu_x$$

$$\Rightarrow \mathcal{E}(\phi, \phi) = 0$$

$$\Rightarrow \frac{1}{2} \sum_{x \in V} \sum_{y \in V} \mu_{xy} (\phi(y) - \phi(x))^2 = 0 \Rightarrow \phi(y) = \phi(x) \quad \forall x, y \in V$$

but $\phi(z)=0 \Rightarrow \phi(x)=0 \forall x \in V$

$\Rightarrow (P, \mu)$ is recurrent from $(R2)$.
□

Techniques: — Laplacian and function space

— Dirichlet forms

Reversible

" $\mu_x P_y = \mu_y P_x$ "

$\mu(V) < \infty \Rightarrow$ Recurrence

$(\mathbb{Z}, \mu) \quad \mu_{n,n+1} = \alpha^{|n|} \quad n \in \mathbb{Z}$

$\sum_n \alpha^n < \infty \Leftrightarrow \alpha < 1$