1.9 Stationary & limiting distributions

Example 1.3 (Revisit) $\{X_n\}_{n\geq 0}$ M.C

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \qquad S = \{0,1\} \qquad (\mu_0, \mu_1)$$

$\alpha \neq 0, \beta \neq 0$

$$\cdot\ p^n_{00} = \frac{\alpha}{\alpha+\beta}\left(1-(\alpha+\beta)\right)^n + \beta/\alpha+\beta$$

$$\cdot\ p^n_{10} = -\beta/\alpha+\beta\left(1-(\alpha+\beta)\right)^n + \beta/\alpha+\beta$$

$$\cdot\ \mathbb{P}(X_n=1) = \alpha/\alpha+\beta + (1-(\alpha+\beta))^n\left(\mu_1 - \alpha/\alpha+\beta\right), \quad \mathbb{P}(X_n=0) = \beta/\alpha+\beta + (1-(\alpha+\beta))^n \times (\mu_0 - \beta/\alpha+\beta)$$

$\cdot\ p^n_{00} \to \beta/\alpha+\beta$ $p^n_{10} \to \beta/\alpha+\beta$
$\cdot\ \mu_1 = \alpha/\alpha+\beta,\ \mu_0 = \beta/\alpha+\beta$ $\Rightarrow X_n \overset{d}{=} \mu$
Stationarity   - limiting   - steady state   - Equilibrium

Definition 1.9.1 A probability π on S is said to be stationary distribution for the Markov chain $\{X_n\}_{n\geq 0}$ with state space S and transition matrix P if

$$\sum_{i\in S} \pi(i)\, p_{ij} = \pi(j) \quad \forall j \in S$$

[Enumerate $S = \{1, .., n\}$ $\quad \pi = [\pi(1) \cdots \pi(n)] \left(\sum_{i=1}^{n} \pi(i) = 1\right)$ $\quad \pi P = \pi$]

Proposition 1.9.2 :- $(X_n)_{n\geq 0}$ is Markov chain with state space S and transition matrix P.

(a) Suppose π is a stationary distribution for the Markov chain

(i) $X_0 \overset{d}{=} \pi \Rightarrow X_n \overset{d}{=} \pi \ \forall n \geq 1$ $\quad$ (ii) $X_0 \overset{d}{=} \pi \Rightarrow (X_{n+m})_{m\geq 0}$ is a MC state space S transition matrix P, initial dist π

(b) $|S| < \infty$ & P is doubly stochastic (i.e. $\sum_{i \in S} p_{ij} = 1 \quad \forall j \in S$)

then $\pi(i) = \frac{1}{|S|} \quad \forall i \in S$ is a stationary distribution.

Proof: (a) (i) $X_0 \overset{d}{=} \pi \Rightarrow \underline{n=1}$ $\mathbb{P}(X_1 = j) = \sum_{i \in S} \mathbb{P}(X_1 = j \mid X_0 = i)\, \mathbb{P}(X_0 = i)$

$$\underset{X_0 \overset{d}{=} \pi}{=} \sum_{i \in S} p_{ij}\, \pi(i) \underset{\pi \text{ is stationary}}{=} \pi(j) \quad \forall j \in S \Rightarrow X_1 \overset{d}{=} \pi$$

Assume $n = k$, $X_k \overset{d}{=} \pi$ then $\mathbb{P}(X_{k+1} = j) = \sum_{i \in S} \mathbb{P}(X_{k+1} = j \mid X_k = i)\, \mathbb{P}(X_k = i)$

$$\underset{\text{Markov Property},\ X_k \overset{d}{=} \pi}{=} \sum_{i \in S} p_{ij}\, \pi(i) = \pi(j) \quad \forall j \in S \Rightarrow X_{k+1} \overset{d}{=} \pi \qquad \square$$

(ii) Ex.

(b) Let π be a Probability on S given by

$$\pi(i) = \frac{1}{|S|} \quad \forall i \in S \quad \& \quad \pi(A) = \sum_{i \in A} \frac{1}{|S|} = \frac{|A|}{|S|}$$

P – doubly stochastic

$$j \in S, \quad \sum_{i \in S} \pi(i)\, p_{ij} = \frac{1}{|S|} \sum_{i \in S} p_{ij} = \frac{1}{|S|} = \pi(j)$$

def of π

$\therefore$ π is stationary for $(X_n)_{n \geq 0}$ $\square$

Example 1.13:- Note $\frac{\beta}{\alpha+\beta} = \pi(0)$ & $\alpha/_{\alpha+\beta} = \pi(1)$

is a stationary distribution for M.C with $P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$

Proposition 1.9.

Then π

Proof:

$|S|$

Proposition 1.9.3 : Let $|S| < \infty$. Let $(X_n)_{n\geq 0}$ be a M.C on S with transition matrix P. Suppose for some $i \in S$

$$p^n_{ij} \to \pi(j) \quad \text{as } n \to \infty \quad \forall j \in S$$

Then π is a stationary distribution for $(X_n)_{n\geq 0}$.

Proof:

- $$\sum_{j\in S} \pi(j) = \sum_{j\in S} \lim_{n\to\infty} p^n_{ij} \underset{|S|<\infty}{=} \lim_{n\to\infty} \sum_{j\in S} p^n_{ij} = 1$$

- $$p^n_{ij} = \sum_{k\in S} p^{n-1}_{ik} p_{kj}$$

$$\Rightarrow \lim_{n\to\infty} p^n_{ij} \underset{|S|<\infty}{=} \sum_{k\in S} \lim_{n\to\infty} p^{n-1}_{ik} p_{kj} \Rightarrow \pi(j) = \sum_{k\in S} \pi(k) p_{kj}$$

$\square$

Example 1.9.4

$S = \{0, 1, 2\}$

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 1/2 & 1/4 \\ 1/6 & 1/3 & 1/2 \end{bmatrix}$$

$(X_n)_{n \geq 0}$ M.C on S with transition matrix P

Suppose π is stationary then:-

① $\dfrac{\pi(0)}{3} + \dfrac{\pi(1)}{4} + \dfrac{\pi(2)}{6} = \pi(0)$

② $\dfrac{\pi(0)}{3} + \dfrac{\pi(1)}{2} + \dfrac{\pi(2)}{3} = \pi(1)$

③ $\dfrac{\pi(0)}{3} + \dfrac{\pi(1)}{4} + \dfrac{\pi(2)}{2} = \pi(2)$

④ $\pi(0) + \pi(1) + \pi(2) = 1$

$\boxed{2 \times ① - ②} \Rightarrow \pi(1) = \frac{5}{3}\pi(0)$

$\boxed{\text{in } ①} \Rightarrow \pi(2) = \frac{3}{2}\pi(0)$

$\boxed{\text{in } ④}$ $\pi(0)\left(1 + \frac{5}{3} + \frac{3}{2}\right) = 1$

$\pi(0) = 6/25$

$\pi(1) = 10/25$

$\pi(2) = 9/25$

$\Rightarrow$ $\boxed{\text{Satisfies } ③}$

Example 1.9.5 : $\{X_n\}_{n\geq 0}$ is a M.C on S with transition matrix P.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}, \quad S = \{1, 2, 3\}$$

π is stationary $\Rightarrow$

$$\left.\begin{aligned} &\tfrac{1}{2}\pi(3) = \pi(1) \\ &\pi(1) + \tfrac{1}{2}\pi(2) = \pi(2) \\ &\frac{\pi(2)}{2} + \frac{\pi(3)}{2} = \pi(3) \\ &\pi(1) + \pi(2) + \pi(3) = 1 \end{aligned}\right\} \underset{(Ex.)}{\Rightarrow} \begin{aligned} &\pi(1) = \tfrac{1}{5}, \\ &\pi(2) = 2/5 \\ &\pi(3) = 2/5 \end{aligned}$$

$$p_{11}^{n} = \frac{1}{5} + \frac{1}{2^n}\left(\frac{4}{5}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{5}\sin\left(\frac{n\pi}{2}\right)\right) \xrightarrow[n\to\infty]{} \frac{1}{5}$$

Summary:

- $|S| < \infty$, Fix $i \in S$, $p_{ij}^{n} \to \pi(j)$, $\forall j \in S \Rightarrow \pi$ is stationary

  π is also called Equilibrium distribution or Steady state distribution for the markov chain

- $|S| < \infty$, $\sum_{i \in S} p_{ij} = 1$ then $\pi(i) = \frac{1}{|S|}$, $i \in S$ is a stationary distribution

Q1: Is π - stationary distribution always unique? [Give in HW]

Q2: Steady state versus stationary?

Away: 17, 19, 24, 26
Take: W 20 → F 22 10-11
27
Oct 4 → F 6 10-11
11 → F 13 10-11

Example: Ehrenfest chain, $d = 3$

$$\begin{tikzcd} 0 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "1/3"] \arrow[r, bend left, "2/3"] & 2 \arrow[l, bend left, "1/3"] \arrow[r, bend left, "1/3"] & 3 \arrow[l, bend left, "1"] \end{tikzcd}$$

$S = \{0, 1, 2, 3\}$

$(X_n)_{n \geq 0}$ is a M.C on S with transition matrix P.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix} = P$$

$$\boxed{\lim_{n \to \infty} p^n_{00} \text{ does not exist}}$$

Solve $\pi P = \pi$

$$\left.\begin{aligned} \frac{\pi(1)}{3} &= \pi(0) \\ \pi(0) + \frac{2}{3}\pi(2) &= \pi(1) \\ \frac{2}{3}\pi(1) + \pi(3) &= \pi(2) \\ \frac{1}{3}\pi(2) &= \pi(3) \\ \pi(0) + \pi(1) + \pi(2) + \pi(3) &= 1 \end{aligned}\right\} \xrightarrow{\text{Ex}} \begin{aligned} \pi(0) &= 1/8 \\ \pi(1) &= 3/8 \\ \pi(2) &= 3/8 \\ \pi(3) &= 1/8 \end{aligned}$$

Example 1.5.4 (Revisited) $(X_n)_{n\geq 0}$ — $a_i \geq 0$ $p_i \geq 0$

Birth death chain

$$a_i + p_i = 1$$

$$\overset{a_i}{\longleftarrow} i \overset{p_i}{\longrightarrow}$$

$$a_0 \circlearrowleft 0 \xrightarrow{p_0} 1$$

π _n stationary $\Rightarrow$

$$\begin{cases} (*)\ \pi(0) = \sum_{i \in S} \pi(i) p_{i0} = \pi(1) a_1 + \pi(0) a_0 \\ j \neq 0 \\ (**)\ \pi(j) = \pi(j-1) p_{j-1} + \pi(j+1) a_{j+1} \end{cases}$$

$j \neq 0$

$$(**) \Rightarrow \pi(j) a_j - \pi(j-1) p_{j-1} = \pi(j+1) a_{j+1} - \pi(j) p_j$$

$$(*) \Rightarrow \pi(1) a_1 - \pi(0) p_0 = 0$$

$$\Rightarrow \pi(j) = \frac{p_{j-1}}{a_j} \pi(j-1) \quad \forall j \neq 0$$

$\Rightarrow$ let $\gamma_j = \prod_{k=1}^{j} \frac{p_{k-1}}{q_k}$

$+ \boxed{\pi(0)\, q_0}$

(induction iteration) $\quad \pi(j) = \gamma_j\, \pi(0) \quad \forall j \neq 0$

π is a Probability $\Rightarrow$

$$\pi(0)\left(\sum_{j=1}^{\infty} \gamma_j + 1\right) = 1$$

If $\sum_{j=1}^{\infty} \gamma_j < \infty$ then $c = 1 + \sum_{j=1}^{\infty} \gamma_j \Rightarrow \begin{cases} \pi(0) = \frac{1}{c} \\ \pi(j) = \gamma_j / c \end{cases}$ is a stationary distribution for $(X_n)_{n \geq 0}$

If $\sum_{j=1}^{\infty} \gamma_j = \infty$ then no stationary distribution exists

$q_0 = 1$ & $p_0 = 1$ (Ex)

Thm 1.9.7 (

chain on

Then $(X_n)_{n \geq}$

Thm 1.9.8 :-

space S, with

Thm 1.9.7 (Markov chain tree Theorem) Let X $(X_n)_{n\geq 0}$ be a Markov chain on state space S $(|S|<\infty)$, with an irreducible transition matrix P. Then $(X_n)_{n\geq 0}$ has a [unique] stationary distribution.

Thm 1.9.8:- $N_n(i) = \sum_{k=1}^{n} \mathbb{1}(X_k = i)$ Let (X_n) be a Markov chain on state space S, with an irreducible transition matrix P.

$$\frac{N_n(i)}{n} \longrightarrow \frac{1}{E_i(T_i)}$$ as $n \to \infty$ with probability 1.