

Exponential distribution

$$P(T > t) = e^{-\lambda t} \quad t \geq 0 \quad T \sim \text{Exp}(\lambda)$$

$$ET = \frac{1}{\lambda}$$

Thm 23.2 Let S be a countable set, and

let $T_k, k \in S$ independent $\text{Exp}(q_k)$ with $0 < q = \sum_{k \in S} q_k < \infty$

let $T = \inf_k T_k$. Then the infimum is attained

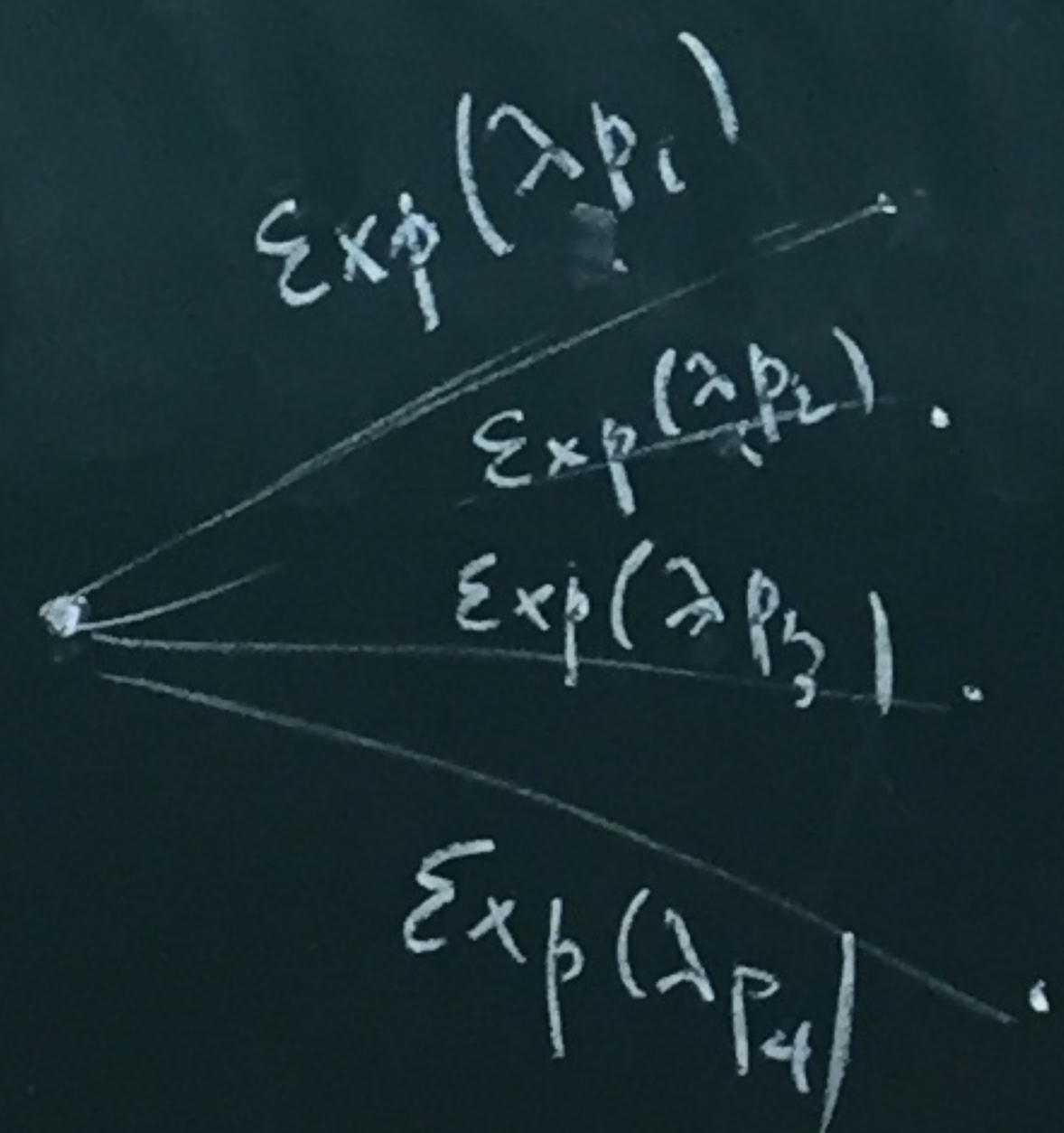
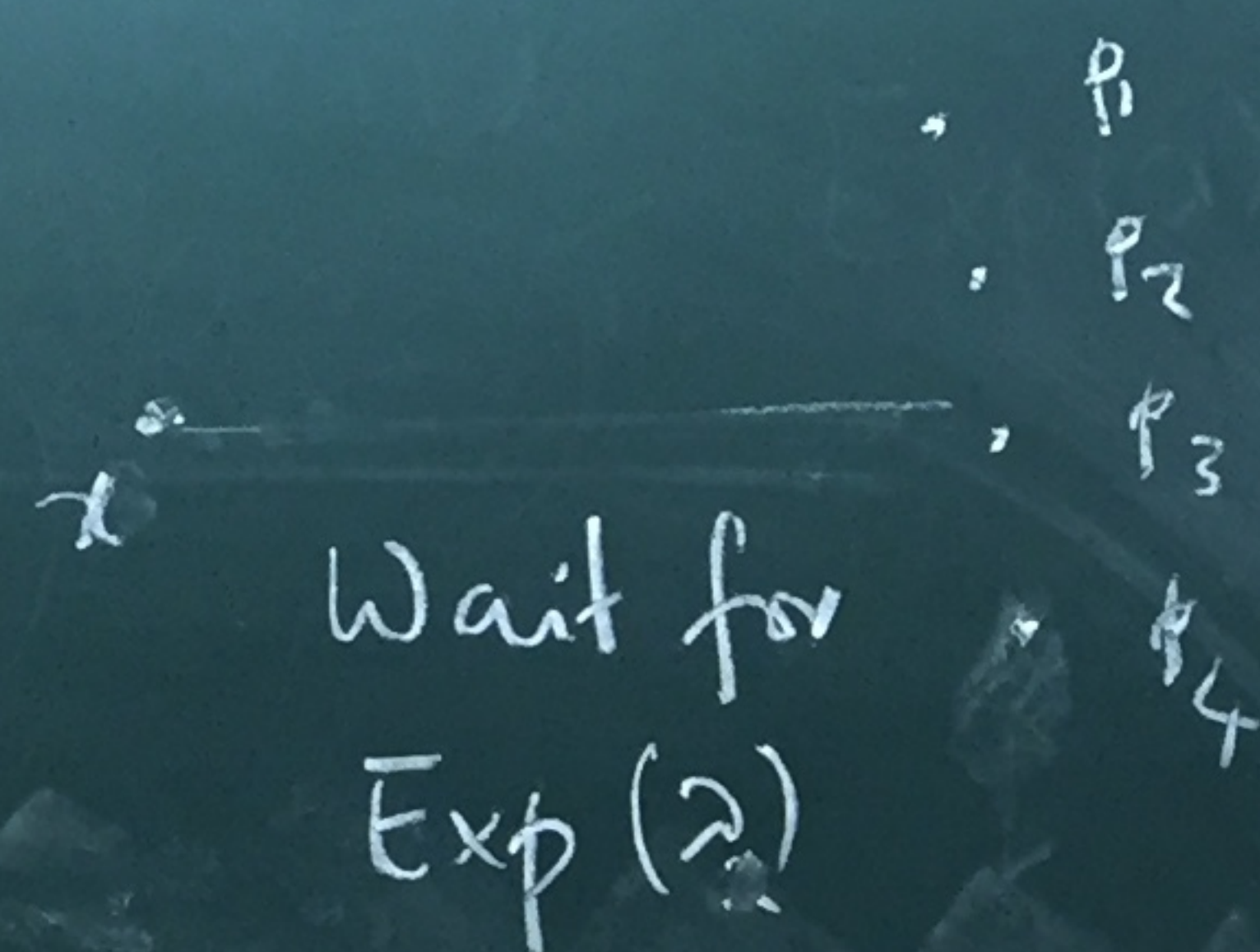
at a unique random value K of k with probability 1.

Moreover, T and K are independent, with $T \sim \text{Exp}(q)$

$$P(K=k) = q_k/q$$

Continued

Continuous time MC



P_f

Set $K = \begin{cases} k & \text{if } T_k < T_j \text{ for all } j \neq k \\ \text{undefined} & \text{o.w.} \end{cases}$

$P(K=k)$

$$\begin{aligned}
 P(T \geq t) &= P(T_i \geq t \text{ for all } i) \\
 &= \prod_{i=1}^{\infty} P(T_i \geq t) = \prod_{i=1}^{\infty} e^{-\lambda_i t} \\
 &= e^{-\sum \lambda_i t}
 \end{aligned}$$

$$P(K=k, T \geq t) = P(T_k \geq t \text{ and } T_j > T_k \text{ for all } j \neq k)$$

$$= \int_t^\infty q_k e^{-q_k s} P(T_j > s \text{ for all } j \neq k) ds$$

$$= \int_t^\infty q_k e^{-q_k s} \prod_{j \neq k} e^{-q_j s} ds$$

$$= q_k \int_t^\infty e^{-q s} ds = \frac{q_k}{q} e^{-q t}$$

Therefore

$$P(K=k) = \frac{q_k}{q}$$

and K & T
are independent.

Remark:

$$\sum q_k < \infty \text{ is}$$

crucial for infimum

to be attained at

unique point; because

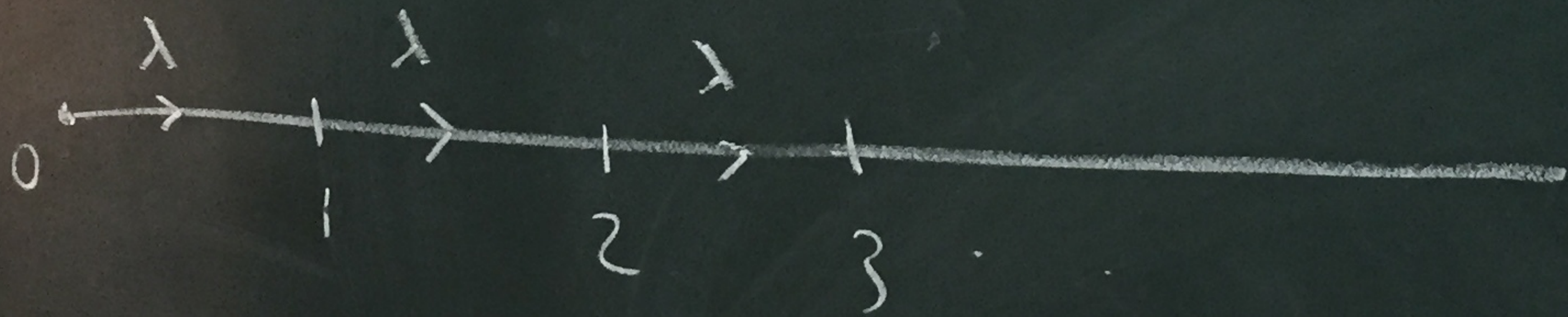
T_k would be large for large k .

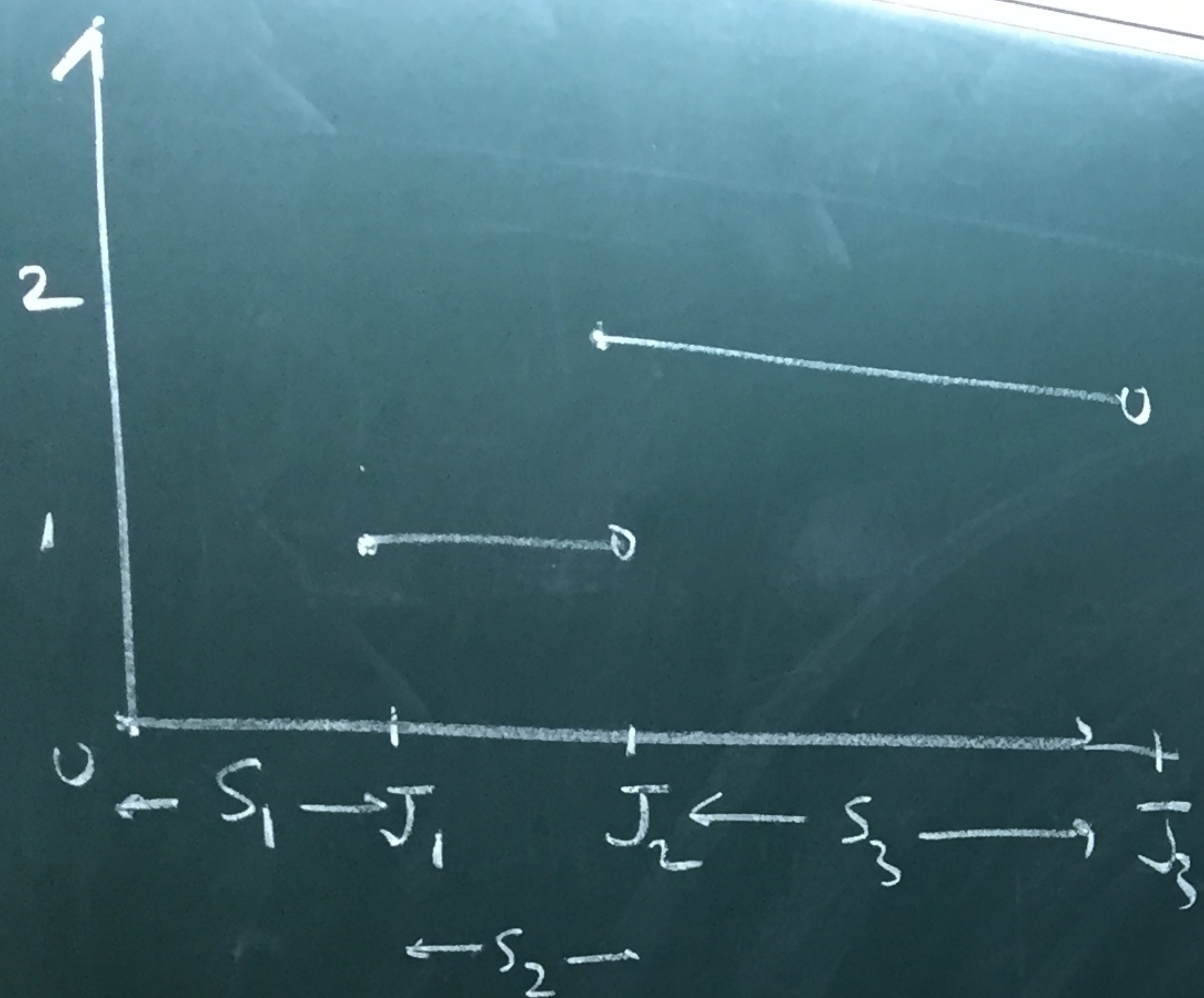
pf

$\mathbb{P}(T$

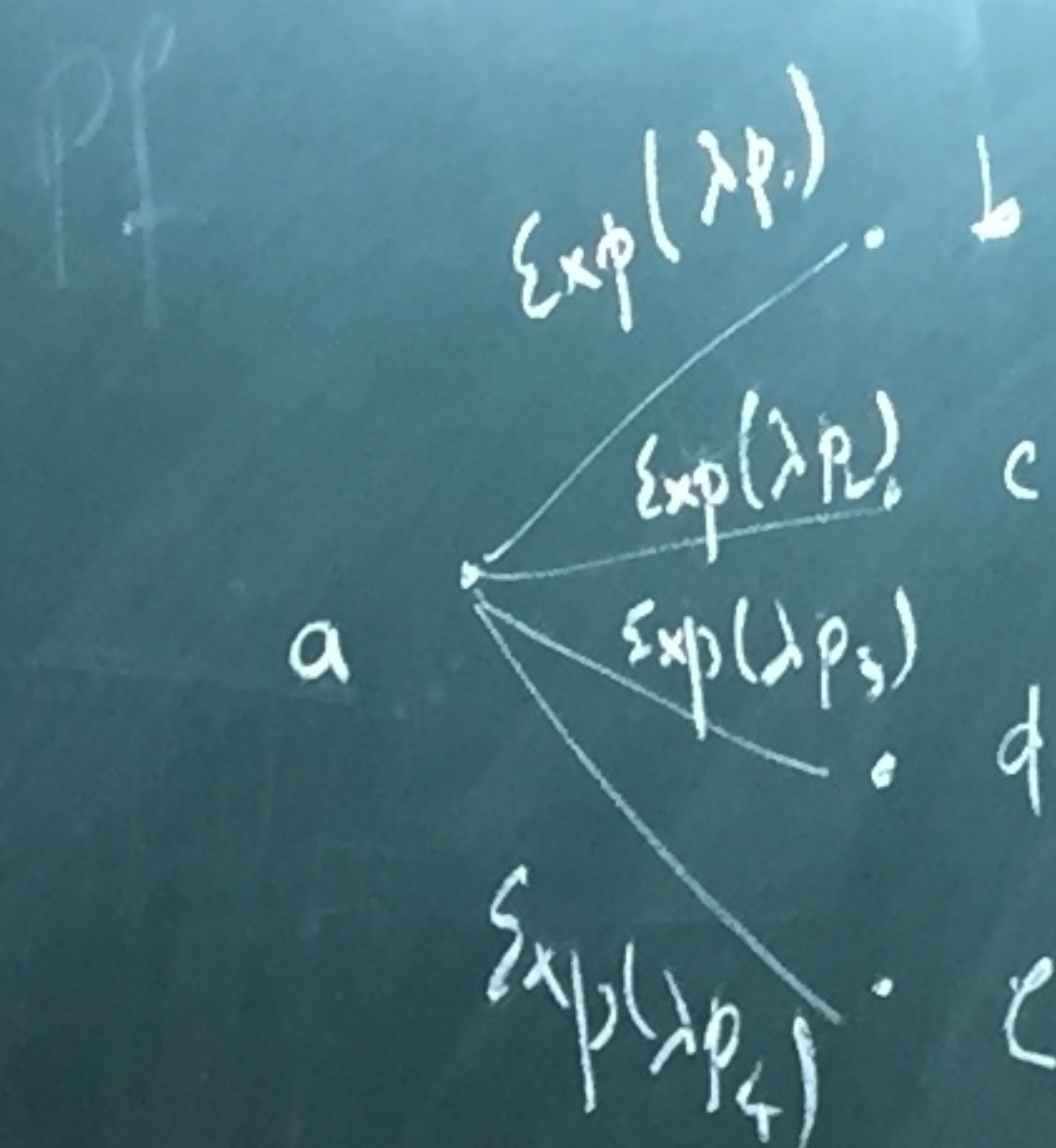
Poisson process

Def A right continuous $(X_t)_{t \geq 0}$ with values in $\{0, 1, 2, \dots\}$ is a Poisson process of rate λ ($0 < \lambda < \infty$) if $S_1, S_2, \dots$ are iid $\text{Exp}(\lambda)$ and jump chain is $Y_n = n$





$$Q = \begin{pmatrix} 0 & -\lambda & \lambda & 0 \\ 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



$$a \begin{pmatrix} b & a & c & d & e \\ \lambda p_1 & -\lambda & \lambda p_2 & \lambda p_3 & \lambda p_4 \end{pmatrix}$$

Remark

Remark. Since $\sum \frac{1}{\lambda_n} = \sum \frac{1}{\lambda} = \infty$

We have $P\left(\sum_{n=1}^{\infty} S_n = \infty\right) = 1$, hence there is no explosion.

Thm. 24.1

Remark:

Construct Poisson process by starting with $S_1, S_2, \dots$ iid $\text{Exp}(\lambda)$

$$\text{Put } J_0 = 0, \quad J_n = S_1 + S_2 + \dots + S_n$$

$$X_t = n \quad \text{if } J_n \leq t < J_{n+1}$$

Thm 24.1 (Markov property) Let $(X_t)_{t \geq 0}$ be Poisson process (λ) .

Then for any $s \geq 0$, $(X_{s+t} - X_s)_{t \geq 0}$ is also a

Poisson process (λ) , independent of $(X_u, u \leq s)$.

Poisson process

an event involving paths.

PI

Need to
show

$$\begin{aligned} \mathbb{P}\left((X_{s+t} - X_s)_{t \geq 0} \in A\right) &= \mathbb{P}\left((X_t)_{t \geq 0} \in A\right) \\ \mathbb{P}\left((X_{s+t} - X_s)_{t \geq 0} \in A, (X_r)_{r \leq s} \in B\right) \\ &= \mathbb{P}\left((X_{s+t} - X_s)_{t \geq 0} \in A\right) \mathbb{P}\left((X_r)_{r \leq s} \in B\right) \end{aligned}$$

Note

The

It is sufficient (since holding times determine Poisson process)

$$\mathbb{P} \left((X_{s+t} - X_s)_{t \geq 0} \in A, \quad X_s = i, \quad (s_1, s_2, \dots, s_n) \in C \right) \\ = \mathbb{P} \left((X_{s+t} - X_s)_{t \geq 0} \in A \right) \mathbb{P} \left(X_s = i, \quad (s_1, s_2, \dots, s_n) \in C \right)$$

$$\tilde{X}_t = X_{t+s} - X_s$$

$$\text{Note } \{X_s = i\} = \{J_i \leq s < J_{i+1}\} = \{J_i \leq s\} \cap \{J_{i+1} > s - J_i\}$$

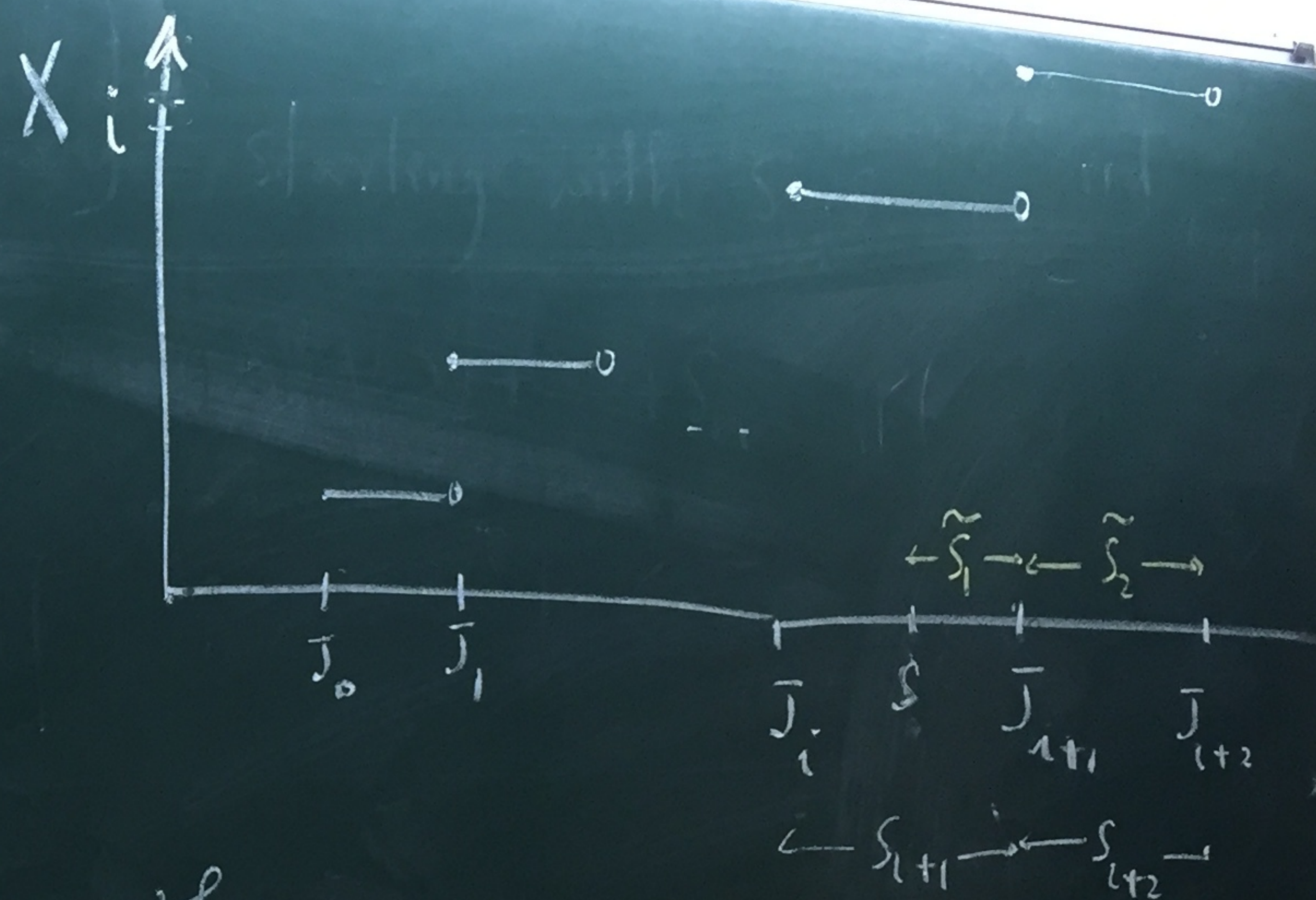
The holding times $\tilde{s}_1, \tilde{s}_2, \dots$ for $\tilde{X}$

$$\tilde{S}_1 = J_{i+1} - s = S_{i+1} - (s - J_i)$$

$$\tilde{S}_n = S_{i+n} \text{ for } n \geq 2$$

Recall that S_1, S_2 are iid $\text{Exp}(\lambda)$

Conditioning on S_i , S_i & $\{X_s = i\}$, by the memoryless property of S_{i+1} & independence, we have $\tilde{S}_1, \tilde{S}_2, \dots$ are $\text{Exp}(\lambda)$



Poisson process

Hence conditional on $\{X_s=1\}$ & $S_1, S_2, \dots, S_n$, with S_i , we have $(\tilde{X}_t)_{t \geq 0}$ is a Poisson process (λ) and is independent of $(X_r, r \leq s)$.

Thm (Strong Markov property): Let $(X_t)_{t \geq 0}$ be a Poisson process (λ) , and let T be a stopping time of $(X_t)_{t \geq 0}$. Then conditional on $T < \infty$, $\{X_{T+t} - X_T\}_{t \geq 0}$ is Poisson process (λ) and is independent of $(X_s, s \leq T)$.

Def T Stopping time for $(X_t)_{t \geq 0}$

$\{T \leq s\}$ depends on $\{X_r, r \leq s\}$

Pf Later.

Def. For $(X_t)_{t \geq 0}$ real valued process

• $X_t - X_s$ is increment over $(s, t]$

• $(X_t)_{t \geq 0}$ has stationary increments

if the distribution of $X_{s+t} - X_s$ depends only on $t \geq 0$

• $(X_t)_{t \geq 0}$ has independent increments if increments over finite collection of disjoint intervals are independent

Thm Let $(X_t)_{t \geq 0}$ be an increasing, right continuous integer valued process starting from 0. Let $0 < \lambda < \infty$.

Then the following conditions are equivalent.

(a) (jump chain / holding time defn). The holding times $S_1, S_2, \dots$ of $(X_t)_{t \geq 0}$ are independent $\text{Exp}(\lambda)$ & $Y_n = n$ for all n

(b) (infinitesimal defn) $(X_t)_{t \geq 0}$ has independent increments and as $h \downarrow 0$,

$$\frac{o(h)}{h} \rightarrow 0 \text{ as } h \downarrow 0$$

uniformly in t we have

$$\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h), \quad \mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h)$$

(c) (transition probability defn)

$(X_t)_{t \geq 0}$ has stationary, independent increments and for each t , $X_t \sim \text{Poisson}(\lambda t)$

Poisson process

an event involving paths

a) $\Rightarrow$ b)

$(a, b]$

$(c, d]$

disjoint, $b \leq c$

$X_d - X_c$ indep of $X_b - X_a$?

use $X_d - X_c$

independent of $\{X_s, s \leq c\}$

$$\begin{aligned} \mathbb{P}(X_{t+h} - X_t \geq 1) &= \mathbb{P}(\tilde{S}_1 \leq h) = 1 - e^{-\lambda h} \\ &= 1 - [1 - \lambda h + o(h)] \\ &= \lambda h + o(h) \end{aligned}$$

$$\begin{aligned} \mathbb{P}(X_{t+h} - X_t \geq 2) &= \mathbb{P}(\tilde{J}_2 \leq h) \\ &\leq \mathbb{P}(\tilde{S}_1 \leq h, \tilde{S}_2 \leq h) = (1 - e^{-\lambda h})^2 = o(h) \end{aligned}$$

$$S_o \quad \mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h)$$

$$\begin{aligned} \mathbb{P}(X_{t+h} - X_t = 0) &= 1 - \mathbb{P}(X_{t+h} - X_t \geq 1) \\ &= 1 - \lambda h + o(h) \end{aligned}$$

c) $\Rightarrow$ b) $\phi_j(t) := \mathbb{P}(X_t = j)$

Then for $j=1, 2, \dots$, $h>0$

$$\begin{aligned} \phi_j(t+h) &= \mathbb{P}(X_{t+h} = j) \\ &= \sum_{i=0}^j \mathbb{P}(X_t = j-i) \mathbb{P}(X_{t+h} - X_t = i) \end{aligned}$$

$$\phi_j(t+h) = (1 - \lambda h + o(h)) \phi_j(t) + (\lambda h + o(h)) \phi_{j-1}(t) + o(h)$$

$$\frac{\phi_j(t+h) - \phi_j(t)}{h} = -\lambda \phi_j(t) + \lambda \phi_{j-1}(t) + \frac{o(h)}{h}$$

Let $h \downarrow 0$ (consider $h \uparrow 0$)

$$\phi_j'(t) = -\lambda \phi_j(t) + \lambda \phi_{j-1}(t) \quad (*)$$

Similarly show $\phi_0'(t) = -\lambda \phi_0(t)$

$$X_0 = 0 \Rightarrow p_0(0) = 1 \quad p_j(0) = 0 \text{ for } j \geq 1$$

Solve (*) using above

$$p_j(t) = e^{-\lambda t} \frac{(\lambda t)^j}{j!}, \quad j = 0, 1, 2, \dots$$

So $X_t \sim \text{Poisson}(\lambda t)$

Note also that for any $s \geq 0$, $\tilde{X}_t = (X_{t+s} - X_s)_{t \geq 0}$ satisfies (b)

Note $\mathbb{P}(\tilde{X}_{t+h} - \tilde{X}_t = 0) = \mathbb{P}(X_{t+s+h} - X_{t+s} = 0)$, so same conditions as (b)

c) $\Rightarrow$ a)

Condition c) determines finite dimensional distribution

$$\mathbb{P}(X_{t_0} = x_0, X_{t_1} = x_1, \dots, X_{t_n} = x_n) \\ = \mathbb{P}(X_{t_0} = x_0, X_{t_1} - X_{t_0} = x_1 - x_0, \\ X_{t_2} - X_{t_1} = x_2 - x_1, \dots)$$

Any finite set

Poisson process

Any right cont. process determined by f.d.d,
including distribn of holding times / jump chain

We showed there is a process satisfying (a) & (c).

Therefore the jump chain / holding time distribution
should be as specified.