Recall

Discrete time Markov chain

S - state space $\quad S - \begin{cases} \text{finite} \\ \text{or} \\ \text{countably infinite} \end{cases}$

P - transition matrix $\quad i, j \in S \quad p_{ij} \in [0,1]$

$\sum_{j \in S} p_{ij} = 1 \quad \forall i \in S$

μ - initial distribution $\quad (S, \mathcal{P}(S))$ - Probability on S

Memo...

Prope...

Pa...

$(\Omega, \mathcal{F}, \mathbb{P})$ $\quad X_n : \Omega \to S \quad \forall n \geq 0$ - state of the system at time n.

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = \mu(\{i_0\})\, p_{i_0 i_1}\, p_{i_1 i_2} \cdots p_{i_{n-1} i_n} \quad \forall n \geq 0$$

X_n - is a Markov chain on state space S, initial distribution μ and transition matrix P.

Memoryless Property :

If $\mathbb{P}(X_{n-1} = i_{n-1}, \dots, X_0 = i_0) > 0$

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, \dots, X_0 = i_0) \quad \forall i_j \in S,\ j \in \{0, \dots, n-1\}$$

$$= \mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}) := p_{i_{n-1} i_n}$$

$\boxed{\text{past}}$ i_{n-1} $\boxed{\text{future}}$

Alitr : $X_n \mid X_{n-1}, \dots, X_0 \overset{d}{=} X_n \mid X_{n-1}$ or $X_{n-2}, \dots, X_0 \mid X_{n-1}$ is independent of $X_n \mid X_{n-1}$

Remarks

(d) $\mathbb{P}(X_n = i) > 0$ then $p_{ij} = \mathbb{P}(X_{n+1} = j \mid X_n = i)$

$\hookrightarrow$ Do not depend on n

Definition 1.21: - $\boxed{\text{Time Homogeneous Markov chain}}$

(e) The entries of P are called one-step transition Probabilities of the Markov chain $(X_n)_{n \geq 0}$. The stochastic matrix P is called one step transition probability matrix of the markov chain $(X_n)_{n \geq 0}$.

(f) $n \geq 0$ P^n = n^{th} power of the matrix for $n \geq 1$

$P^0 = I$

Let p^n_{ij} be the $(i,j)^{th}$ entry of P^n

$$p^n_{ij} = \sum_{\substack{i_k \in S \\ k=1,\dots,n-1}} p_{ii_1} p_{i_1 i_2} \cdots p_{i_{n-1} j}$$

(Obvious from definition of matrix multiplication in $|S|<\infty$ case. Ex S- countable)

$$= \sum_{\substack{i_k \in S \\ k=1,\dots,n-1}} \frac{\mu(\{i\})\, p_{ii_1} p_{i_1 i_2} \cdots p_{i_{n-1} j}}{\mu(\{i\})} = \sum_{\substack{i_k \in S \\ k=1,\dots,n-1}} \frac{\mathbb{P}(X_0=i, X_1=i_1, \dots, X_{n-1}=i_{n-1}, X_n=j)}{\mu(\{i\})}$$

$$= \frac{\mathbb{P}(X_0 = i,\ X_n = j)}{\mathbb{P}(X_0 = i)} = \mathbb{P}(X_n = j \mid X_0 = i)$$

(g)

$p_{ij}^n \equiv$ Probability that the chain will be at state j in 'n' steps starting from state i

$P^n \equiv$ n^{th} step transition Probability matrix for the Markov chain $(X_n)_{n \geq 0}$

(g) Chapman - Kolmogorov Equations

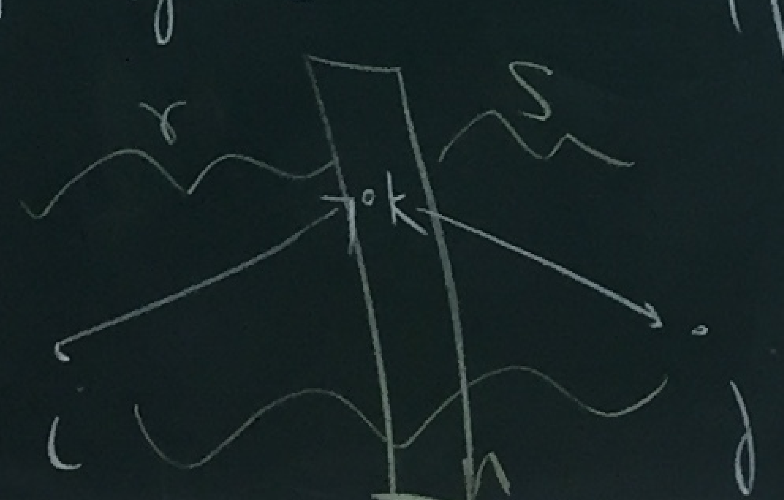
$n \in \mathbb{N}$ $r \in \mathbb{N}$ $s \in \mathbb{N}$ $r+s=n$

$$P^n = P^{r+s} = P^r P^s$$

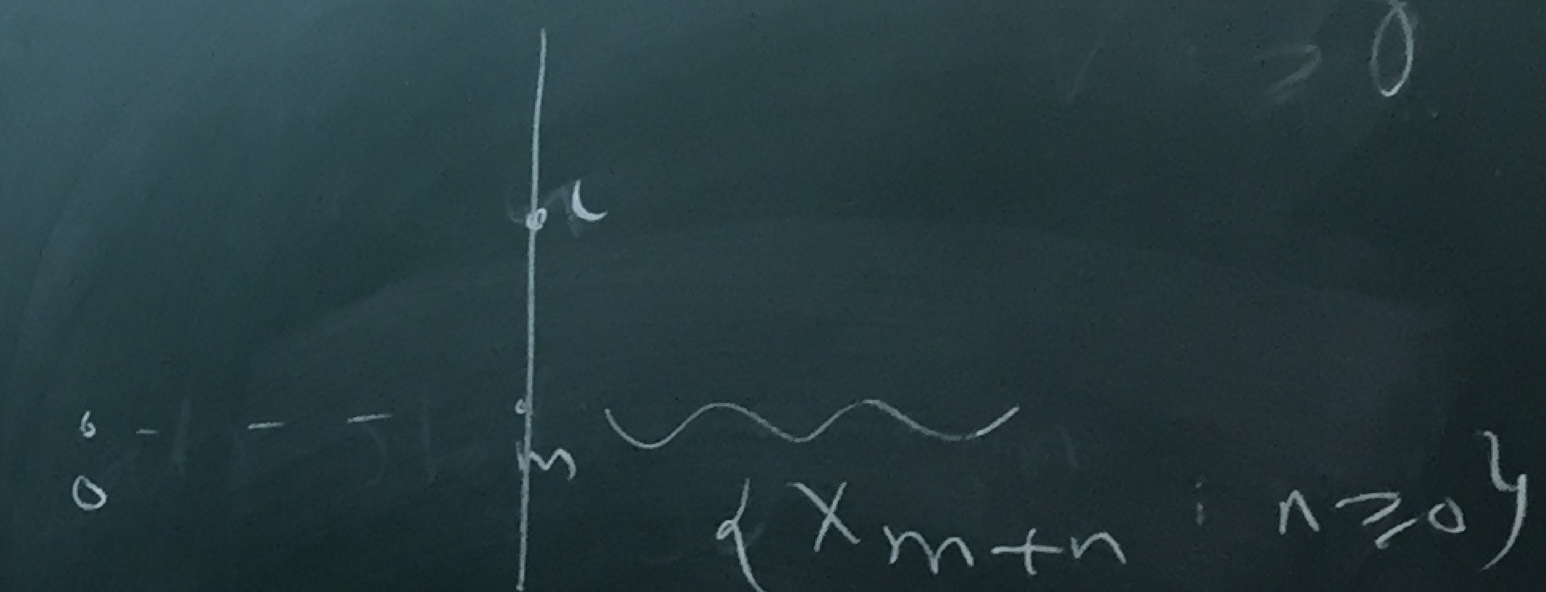
obvious $|S|<\infty$
- Countable case (Ex)

$$\Rightarrow \forall i,j \in S \quad p_{ij}^n = \sum_{k \in S} p_{ik}^r p_{kj}^s$$

$$\Rightarrow \forall i,j \in S \quad \mathbb{P}(X_n = j \mid X_0 = i) = \sum_{k \in S} \mathbb{P}(X_r = k \mid X_0 = i)\, \mathbb{P}(X_s = j \mid X_0 = k)$$



Markov Property: $\{X_n\}_{n \geq 0}$ be Markov with state space S, initial distribution μ & transition matrix P.



$m \geq 0$, $\{X_{m+n}\}_{n \geq 0}$ condition on $X_m = i$

(i) $\{X_{m+n}\}_{n \geq 0} \mid X_m = i$ is a Markov chain on state space S, transition matrix P, & initial distribution δ_i.

(ii) $\{X_{m+n}\}_{n \geq 0} \mid X_m = i$ is independent of $X_0, \ldots, X_{m-1} \mid X_m = i$

distribution

≥ 0

$n \geq 0\}$

1.3 Examples

$0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 \leq \mu_0, \mu_1 \leq 1$

$\mu_0 + \mu_1 = 1$

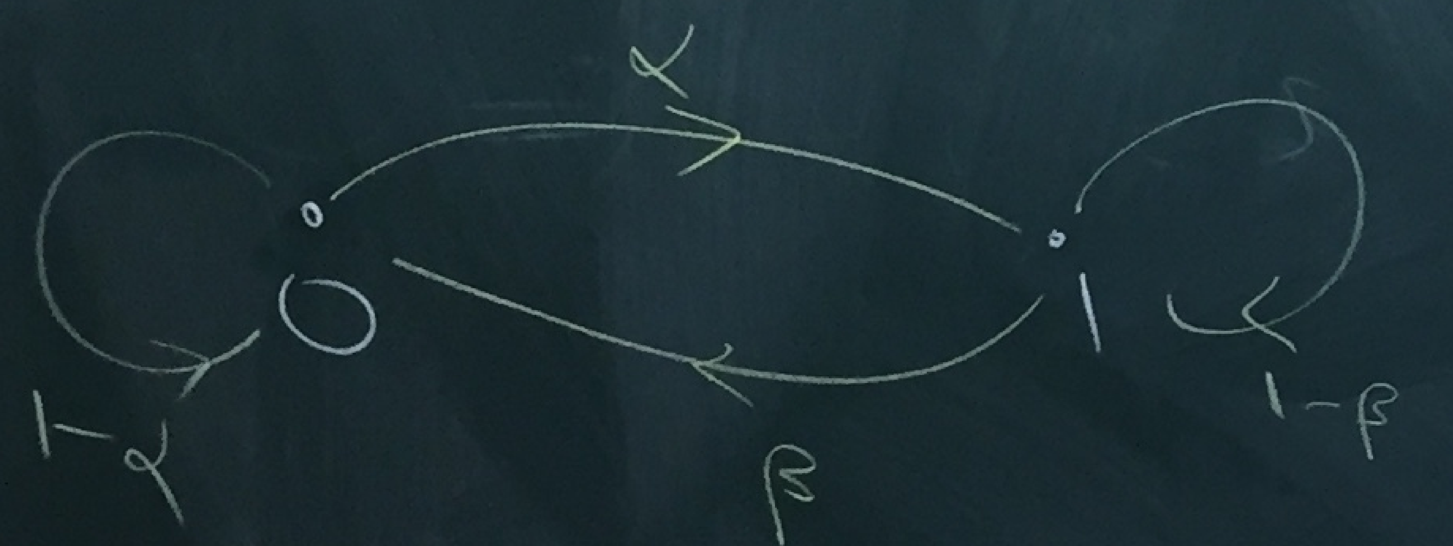
Example 1.3.1: $S = \{0,1\}$ $\quad P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$

$\mu_0 = \mu(\{0\})$

$\mu_1 = \mu(\{1\})$

$\{X_n\}_{n \geq 0}$ is a Markov chain on (S, P, μ).

Graph



$$\mathbb{P}(X_0 = x_0, \ldots X_n = x_n) = \mu(\{x_0\})\, \alpha^{n_{01}} \beta^{n_{10}} (1-\alpha)^{n_{00}} (1-\beta)^{n_{11}}$$

$n_{ij} \equiv$ # of one step transitions from i to j in $X_0 = x_0, \ldots, X_n = x_n$

$x_i \in \{0,1\}$

$$P^{n+1} = P^n P$$

$$\Rightarrow p^{n+1}_{00} = p^n_{00}(1-\alpha) + p^n_{01}\beta$$

$$p^n_{00} + p^n_{01} \Rightarrow \boxed{p^{n+1}_{00} = p^n_{00}(1-\alpha-\beta) + \beta}$$

$$\|$$

$$\mathbb{P}(X_n=0 \mid X_0=0) + \mathbb{P}(X_n=1 \mid X_0=0)$$

$$\|$$

$$\mathbb{P}(X_n=0 \cup X_n=1 \mid X_0=0) = 1$$

convention

$$p^0_{00} = 1$$

Recurrence

$$x_n =$$

$$x_{n+1} =$$

Constant s

$$\Rightarrow x$$

$$\Rightarrow (\alpha$$

$$\Rightarrow (\alpha+\beta \neq 0)$$

Recurrence Equations

$x_n = p^n_{00}$, $x_0 = 1$

$x_{n+1} = (1-\alpha-\beta) x_n + \beta \quad \forall n \geq 0$

Constant solution : $x_n = x \quad \forall n \geq 1$

$\Rightarrow x = (1-\alpha-\beta) x + \beta$

$\Rightarrow (\alpha+\beta) x = \beta$

$\Rightarrow x = \beta / \alpha+\beta$

$(\alpha+\beta \neq 0)$

Set $y_n = x_n - x$

$= x_{n-1}(1-\alpha-\beta) + \beta - \beta/\alpha+\beta$

$= (y_{n-1} + \beta/\alpha+\beta)(1-(\alpha+\beta))$

$+ \beta - \beta/\alpha+\beta$

$\Rightarrow y_n = y_{n-1}(1-(\alpha+\beta))$

$\Rightarrow y_n = (1-(\alpha+\beta))^n y_0$

$\Rightarrow y_n = (1-(\alpha+\beta))^n (1 - \beta/\alpha+\beta)$

$\Rightarrow$ $x_n - \beta/_{\alpha+\beta} = (1-(\alpha+\beta))^n (1 - \beta/_{\alpha+\beta})$

$$x_n = \frac{\alpha}{\alpha+\beta}(1-(\alpha+\beta))^n + \beta/_{\alpha+\beta}$$

$$p^n_{00} = \begin{cases} \frac{\alpha}{\alpha+\beta}(1-(\alpha+\beta))^n + \beta/_{\alpha+\beta} & \alpha+\beta \neq 0 \\ 1 & \alpha+\beta = 0 \end{cases}$$

$$p_{10}^{n+1} = p_{10}^{n}(1-\alpha) + p_{11}^{n}\beta \qquad \left(\text{form } P^{n+1} = P^{n}P \right)$$

$$\Rightarrow \boxed{p_{10}^{n+1} = p_{10}^{n}(1-(\alpha+\beta)) + \beta} \qquad p_{10}^{0} = 0$$

let $z_n = p_{10}^{n}$, $z_0 = 0$

$$z_{n+1} = (1-(\alpha+\beta))z_n + \beta$$

Repeat analysis for x_n, to get

$$p_{10}^{n} = (1-(\alpha+\beta))^{n}\left(-\beta/_{\alpha+\beta}\right) + \beta/_{\alpha+\beta} \qquad (\alpha+\beta \neq 0)$$

$$= \beta/_{\alpha+\beta}\left(1-(1-(\alpha+\beta))^{n}\right)$$

$P($

$(\alpha+\beta \neq$

$\alpha+\beta \neq$

$$\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 0, X_0 = 0) + \mathbb{P}(X_n = 0, X_0 = 1)$$

$$= \mathbb{P}(X_n = 0 \mid X_0 = 0)\,\mathbb{P}(X_0 = 0) + \mathbb{P}(X_n = 0 \mid X_0 = 1)\,\mathbb{P}(X_0 = 1)$$

$$= p^n_{00}\,\mu_0 + p^n_{10}\,\mu_1$$

$(\alpha + \beta \neq 0)$

$$= \left(\frac{\alpha}{\alpha+\beta}(1-(\alpha+\beta))^n + \beta/_{\alpha+\beta}\right)\mu_0 + \frac{\beta}{\alpha+\beta}\left(1 - (1-(\alpha+\beta))^n\right)(1-\mu_0)$$

$\boxed{\alpha + \beta \neq 0}$

$$\boxed{\mathbb{P}(X_n = 0) = \beta/_{\alpha+\beta} + (1-(\alpha+\beta))^n\left(\mu_0 - \beta/_{\alpha+\beta}\right)}$$

$$\boxed{\mathbb{P}(X_n = 1) = \frac{\alpha}{\alpha+\beta} + (1-(\alpha+\beta))^n \left(\mu_1 - \frac{\alpha}{\alpha+\beta}\right)}$$

- $n \longrightarrow \infty$ $\quad \mathbb{P}(X_n = 0) \longrightarrow \beta/\alpha+\beta$ & $\mathbb{P}(X_n = 1) \longrightarrow \alpha/\alpha+\beta$

$|1-(\alpha+\beta)| < 1$

$$\begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Stationary distribution?

- $\mu_0 = \beta/\alpha+\beta$ $\Longrightarrow$ $\mathbb{P}(X_n = 0) = \beta/\alpha+\beta$ & $\mathbb{P}(X_n = 1) = \alpha/\alpha+\beta$

$\mu_1 = \alpha/\alpha+\beta$