

Recall:-

$\{X_n\}_{n \geq 0}$ is a Markov chain on state space S ,
transition matrix P and initial distribution μ .

$i \in S$ leads to $j \in S$ if $P_i(X_n = j \text{ for some } n \geq 0) > 0$

$\exists n \geq 0$ such that $p_{ij}^n > 0$
 $i \rightarrow j$

$i \in S$ communicates with j if $i \leftrightarrow j$

$(S, \leftrightarrow)$ equivalence relation, (communicating class)

$C - \text{class}$

$i \in C$

$i \in S$

$S -$

$A \subseteq S$

"inf $\phi = \infty$ "

Example 15.1

$A = \{$

C - communicating class in S
 $i \in C$ and $i \rightarrow j \Rightarrow j \in C$ then C is called closed communicating class.

$i \in S$ is said to be absorbing if $\{i\}$ is a closed communicating class.

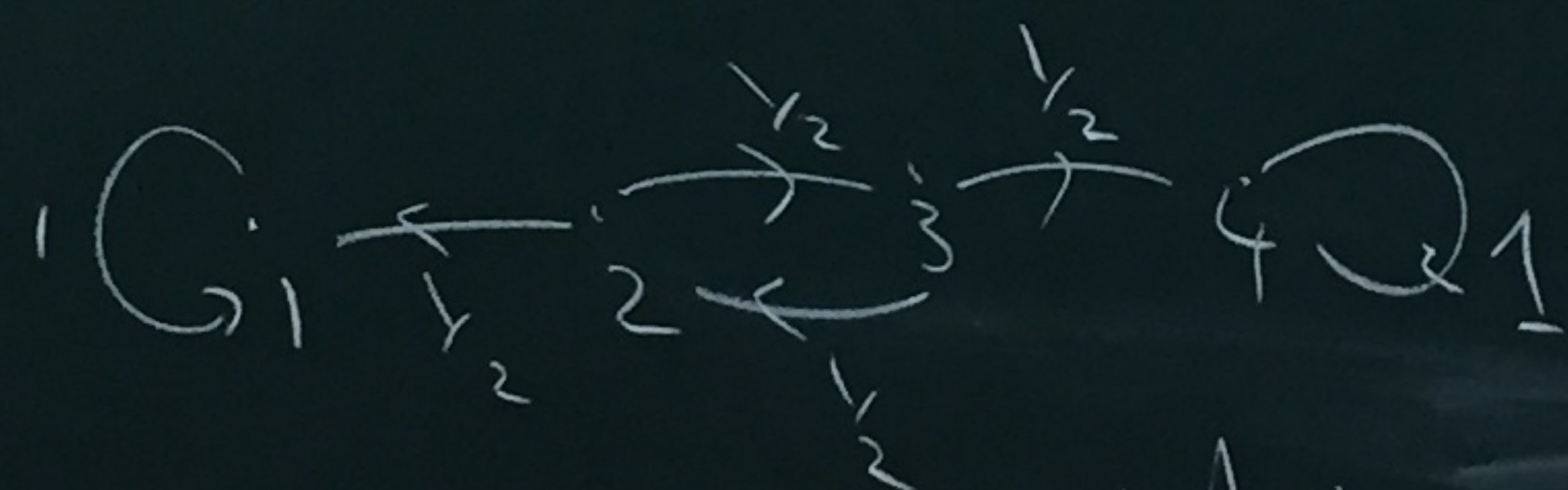
S - one communicating class then $\{X_n\}_{n \geq 0}$ is irreducible.
 $(\forall i, j \in S \exists n \geq 0 \quad p_{ij}^n > 0)$

$$A \subseteq S$$

"inf $\phi = \infty$ "

Example 1.5.1

$$A = \{4\}$$



$$h^A(i) = P_i(T^A < \infty)$$

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$$\begin{cases} h^A(1) = 0, h^A(4) = 1 \\ h^A(2) \stackrel{\text{intuitively}}{=} \frac{1}{2} h^A(1) + \frac{1}{2} h^A(3) \\ h^A(3) = \frac{1}{2} h^A(2) + \frac{1}{2} h^A(4) \end{cases}$$

$$\Rightarrow h^A(2) = \frac{1}{3}, h^A(3) = \frac{2}{3}$$

$$P_4(T^A < \infty) = P_4(T^A = 0) = 1$$

Theorem

(*)

Proof

For

Theorem 1.5.2

$$A \subseteq S$$

let $h^A: S \rightarrow [0,1]$

given by

$$h^A(i) = P_i(T^A < \infty)$$

Then h is the minimal^(*) non-negative solution of the linear

System of Equations

$$(1.5.3) \quad \begin{cases} h^A(i) = 1 & \text{if } i \in A \\ h^A(i) = \sum_{j \in S} p_{ij} h^A(j) & \text{if } i \notin A \end{cases}$$

(*) - minimal $f: S \rightarrow [0,1]$ & f - satisfies (1.5.3) then $f(i) \geq h(i) \forall i \in S$

Proof:- clearly $i \in A \Rightarrow 1 = P_i(T^A = 0) = P_i(T^A < \infty) = h^A(i)$

$$\text{For } i \notin A \quad P_i(T^A \geq 1) = 1$$

$$P_i(T^A=1) = \sum_{j \in A} P_i(X_1=j) = \sum_{j \in A} p_{ij} \quad \text{--- } \textcircled{xx}$$

$$\underline{k \geq 1} \quad P_i(T^A=k) = \sum_{j \in S \setminus A} P_i(T^A=k, X_1=j)$$

$$\boxed{P(T^A=k, X_1 \in A) = 0}$$

$$= \sum_{j \in S \setminus A} \sum_{\substack{l_2 \in S \setminus A \\ l_k \in A \\ 2 \leq l \leq k-1}} P_i(X_1=j, X_2=l_2, \dots, X_{k-1}=l_{k-1}, X_k=l_k)$$

$$\stackrel{+}{=} \sum_{j \in S \setminus A} \sum_{\substack{l_2 \in S \setminus A \\ 2 \leq l \leq k-1 \\ l_k \in A}} p_{ij} p_{jl_2} p_{l_2 l_3} \dots p_{l_{k-1} l_k} = \sum_{j \in S \setminus A} p_{ij} \left(\sum_{\substack{l_2 \in S \setminus A \\ 2 \leq l \leq k-1 \\ l_k \in A}} p_{jl_2} p_{l_2 l_3} \dots p_{l_{k-1} l_k} \right)$$

observe,

$$P_i(T^A = k-1) \stackrel{\text{imitating}}{=} \sum_{\substack{j \in S \setminus A \\ k-1}} p_{ij} \sum_{\substack{l_2 \in S \setminus A \\ 3 \leq l \leq k-1 \\ l_k \in A}} p_{il_2} \dots p_{il_{k-1}} p_{il_k}$$

$$\forall k \geq 1 \quad P_i(T^A = k) = \sum_{j \in S \setminus A} p_{ij} P_j(T^A = k-1) \quad \text{--- (xxx)}$$

$i \notin A$

$$h(i) = P_i(T_A < \infty) = P_i\left(\bigcup_{k=1}^{\infty} \{T^A = k\}\right)$$

$$= \sum_{k=1}^{\infty} P_i(T^A = k) = P_i(T^A = 1) + \sum_{k=2}^{\infty} P_i(T^A = k)$$

Using $\text{(xxx)} = \sum_{j \in A} p_{ij} + \sum_{k=2}^{\infty} \left(\sum_{j \in S \setminus A} p_{ij} P_j(T^A = k-1) \right)$

show:-
 $\left[P_i(T^A < \infty) = \sum_{j \in S} p_{ij} P_j(T^A < \infty) \right]$
 $P_i(T^A < \infty | X_1 = j) = P_j(T^A < \infty)$

Real
Analysis

$$\sum_{j \in A} p_{ij} + \sum_{j \in S/A} p_{ij} \sum_{k=2}^{\infty} P_j(T^A = k-1)$$

$$= \sum_{j \in A} p_{ij} + \sum_{j \in S/A} p_{ij} \sum_{k=1}^{\infty} P_j(T^A = k)$$

$$= \sum_{j \in A} p_{ij} h^A(j) + \sum_{j \in S/A} p_{ij} h^A(j)$$

$$\left[\begin{aligned} h^A(j) &= P_j(T^A < \infty) = \sum_{k=1}^{\infty} P_j(T^A = k) \\ h^A(j) &= 1 \quad \forall j \in A \end{aligned} \right]$$

$$= \sum_{j \in S} p_{ij} h^A(j)$$

$$\Rightarrow i \notin A \quad h^A(i) = \sum_{j \in S} p_{ij} h^A(j). \quad \text{So } h \text{ satisfies (1.53).}$$

Suppose $f: S \rightarrow [0,1]$ is another solution satisfying $\star$
 (1.5.3)

$$i \in A \quad f(i) = 1 \quad \Rightarrow \quad f(i) = h^A(i) \quad \forall i \in A$$

$$f(i) = \sum_{j \in S} p_{ij} f(j)$$

$$= \sum_{j \in A} p_{ij} f(j) + \sum_{j \in S \setminus A} p_{ij} f(j)$$

$$\stackrel{i \notin A}{=} \sum_{j \in A} p_{ij} + \sum_{j \in S \setminus A} p_{ij} f(j) \quad \text{--- } \textcircled{+}$$

$$\Rightarrow f(i) = \mathbb{P}_i(X_1 \in A) + \sum_{j \in S \setminus A} p_{ij} f(j) \quad \text{--- step 1 induction}$$

Iterate in $\oplus$ to get

$$\begin{aligned}
 \mathbb{1}_{A^c} f(i) &= \mathbb{P}_i(X_1 \in A) + \sum_{j \in S/A} p_{ij} \left(\sum_{i_2 \in A} p_{ji_2} + \sum_{i_2 \in S/A} p_{ji_2} f(i_2) \right) \\
 &= \mathbb{P}_i(X_1 \in A) + \sum_{j \in S/A} p_{ij} \sum_{i_2 \in A} p_{ji_2} + \sum_{j \in S/A} p_{ij} \sum_{i_2 \in S/A} p_{ji_2} f(i_2) \\
 &= \mathbb{P}_i(X_1 \in A) + \mathbb{P}_i(X_2 \in A, X_1 \notin A) + \sum_{j \in S/A} p_{ij} \sum_{i_2 \in S/A} p_{ji_2} f(i_2) \\
 &= \mathbb{P}_i(T^A = 1) + \mathbb{P}_i(T^A = 2) + \sum_{j \in S/A} \sum_{i_2 \in S/A} p_{ij} p_{ji_2} f(i_2) \\
 &= \mathbb{P}_i(T^A \leq 2) + \sum_{j \in S/A} \sum_{i_2 \in S/A} p_{ij} p_{ji_2} f(i_2)
 \end{aligned}$$

(Induction)
Ex

$$\begin{array}{|l} f \geq 0 \\ p_{ij} \geq 0 \\ \forall i, j \in S \end{array}$$

$$i \notin A \quad f(i) = \mathbb{P}_i(T^A \leq n) + \sum_{\substack{j \in S/A \\ i_2 \in S/A \\ 2 \leq k \leq n-1}} p_{ji_2} p_{i_2 i_3} \cdots p_{i_{n-2} i_{n-1}} f(i_{n-1}) \quad \forall n \geq 1$$

$$\Rightarrow f(i) \geq \mathbb{P}_i(T^A \leq n) \quad \forall n \geq 1$$

let $n \rightarrow \infty$ on both sides

$$\{T^A \leq n\} \uparrow \{T^A < \infty\} \Rightarrow f(i) \geq \mathbb{P}_i(T^A < \infty) \quad \forall i \notin A$$

$$\Rightarrow f(i) \geq h^A(i) \quad \forall i \notin A$$

□

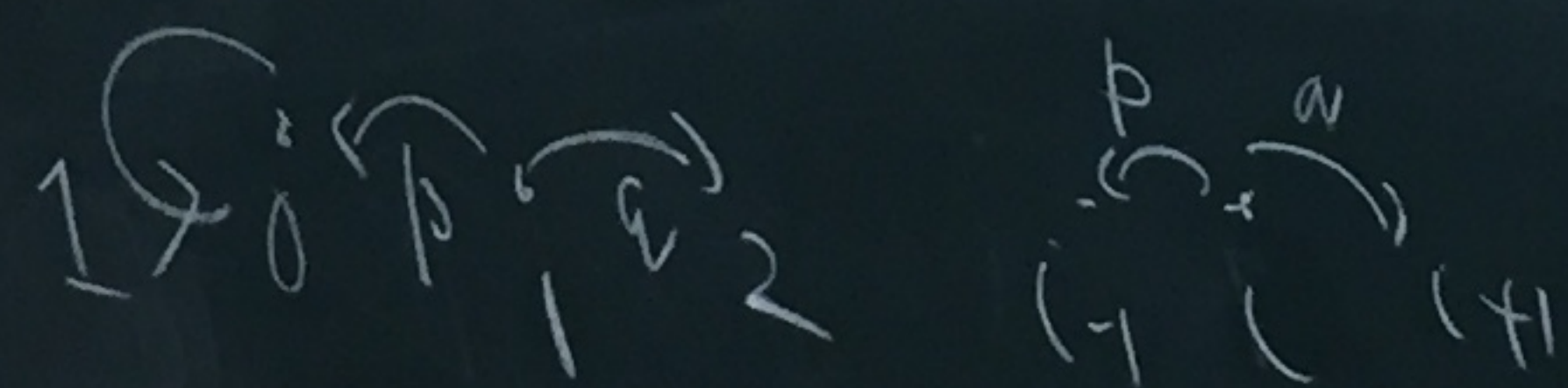
Remark - Example 1.5.1 solution is rigorous.

Example 1.54 [Gambler's Ruin]

- You are in Casino, start with i rupees.
- At each time step, toss a coin with probability of heads $\equiv p$.
 - pay 1 rupee to casino if head appears
 - get 1 rupee from casino if tail appears

$X_n \equiv$ amount of wealth that you have

$\{X_n\}_{n \geq 0}$ Markov chain on $S = \{0\} \cup \mathbb{N}$, $P = [P_{ij}]$



$$\begin{cases} P_{00} = 1, P_{i, i-1} = p, P_{i, i+1} = q, & i \neq 0 \\ P_{ij} = 0, & \forall j \neq i, i-1, i+1 \end{cases}$$

$$A = \{0\} \quad h^A: S \rightarrow [0,1] \quad h^A(i) = \mathbb{P}_i(T^A < \infty)$$

$$\text{Theorem 1.5.2} \Rightarrow h^A(0) = 1$$

$$(i \neq 0, \quad h^A(i) = \sum_{j \in S} p_{ij} h^A(j) = p h^A(i-1) + q h^A(i+1))$$

Recurrence Relation

$$a, b, c \neq 0 \quad a x_{n+1} + b x_n + c x_{n-1} = 0$$

$$\text{let } \lambda \in \mathbb{C} \quad x_n = \lambda^n \quad \text{then} \quad a \lambda^{n+1} + b \lambda^n + c \lambda^{n-1} = 0$$

$$\lambda^{n-1} (a \lambda^2 + b \lambda + c) = 0$$

$$\text{let } \alpha, \beta \text{ be the roots of } a \lambda^2 + b \lambda + c = 0$$

Real
Induction
 $\alpha \neq \beta$
($b^2 - 4ac \neq 0$)

Then $y_n = A\alpha^n + B\beta^n$

$A, B \in \mathbb{C}$

y is a solution
($ay_{n+1} + by_n + cy_{n-1} = 0$)

Suppose we are given x_0 & x_1 ,

Find A, B :
$$\begin{cases} x_0 = A\alpha^0 + B\beta^0 \\ x_1 = A\alpha + B\beta \end{cases} \Rightarrow \begin{bmatrix} A = \frac{\beta x_0 - x_1}{\alpha - \beta}, & B = \frac{\alpha x_0 - x_1}{\beta - \alpha} \end{bmatrix}$$

If $\{z_n\}_{n \geq 1}$ is any other solution with $z_0 = x_0, z_1 = x_1$

Then $a(y_{n+1} - z_{n+1}) + b(y_n - z_n) + c(y_{n-1} - z_{n-1}) = 0 \quad \forall n \geq 1$

As $y_0 = z_0 = x_0$
 $y_1 = z_1 = x_1$ induction implies $y_n = z_n \quad \forall n \geq 1$

$$\boxed{\alpha = \beta}$$

$$y_n = (A + nB)\alpha^n \text{ for some } A, B \text{ is}$$

a general solution (Verify $ay_{n+1} + by_n + cy_{n-1} = 0$)
using $b^2 - 4ac = 0$

Solve for A, B : $x_0 = A\alpha^0, x_1 = (A+B)\alpha$

- Uniqueness is similar as in previous case.

Back to our Example 1.5.4.

$$q h^A(i+1) - h^A(i) + p h^A(i-1) = 0 \quad i \geq 1$$

$$\left. \begin{array}{l} a=q, b=-1, c=p \\ p+q=1 \end{array} \right\}$$

$$\Rightarrow b^2 - 4ac = 1 - 4pq = (p-q)^2$$

$p \neq q \Rightarrow$ distinct roots, $p = q = \frac{1}{2} \Rightarrow$ one root

Case 1: $p \neq q$

Solution

$$h^A(0) = 1 =$$

Case 1: $p \neq q$ $p+q=1$

Roots are
$$\alpha = \frac{1 + \sqrt{1-4pq}}{2q} = \frac{1 + (p-q)}{2q} = p/q$$

$$\beta = \frac{1 - \sqrt{1-4pq}}{2q} = \frac{1 - (p-q)}{2q} = \frac{2q}{2q} = 1$$

Solution

$$h^A(i) = c_2 \alpha^i + c_1 \beta^i \quad \text{for some constants } c_1, c_2$$

$$h^A(i) = c_1 + c_2 \left(\frac{p}{q}\right)^i$$

$$h^A(0) = 1 \Rightarrow c_1 + c_2 = 1 \Rightarrow h^A(i) = \left(\frac{p}{q}\right)^i + c_1 \left(1 - \left(\frac{p}{q}\right)^i\right) \quad i \in S$$

• $p > q$

$$0 \leq h^A(i)$$

• $p < q$

$$0 \leq h^A(i)$$

• $p = q, i = p = 1$

$$h^A(0) = 1 \Rightarrow 1 + c_1 = 1 \Rightarrow c_1 = 0$$

= one not

$$\bullet \quad p > q, \quad (p/q \geq 1) \Rightarrow h^A(i) = (1-c_1)(p/q)^i + c_1$$

$$0 \leq h^A(i) \leq 1 \Rightarrow c_1 = 1 \quad \& \quad h^A(i) = 1 \quad \forall i \in S$$

$$[1-c_1 > 0 \Rightarrow h^A(i) > 1 \text{ for some } i]$$

$$1-c_1 < 0 \Rightarrow h^A(i) < 0 \text{ for some } i$$

$$\bullet \quad p < q, \quad h^A(i) = c_1 + c_2 \left(\frac{p}{q}\right)^i + c_1 \left(1 - \left(\frac{p}{q}\right)^i\right)$$

$$0 \leq h^A(i) \leq 1 \Rightarrow c_1 \geq 0 \quad (\text{as } i \rightarrow \infty, (p/q)^i \rightarrow 0 \text{ and } h^A(i) \rightarrow c_1)$$

Minimal solution will happen at $c_1 = 0$.

$$\bullet \quad p = q, \quad \alpha = \beta = 1 \quad h^A(i) = A + iB$$

$$h^A(i) = 1 \quad \forall i \geq 0$$

$$\Rightarrow h^A(i) = \left(\frac{p}{q}\right)^i$$

$$0 \leq h^A(i) \leq 1 \Rightarrow B = 0$$

$$h^A(0) = 1 \Rightarrow A = 1$$